INFERENCE BASED ON BOUNDARY CROSSING OF DIFFUSIONS

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Nowadays, the boundary crossing problem of diffusion processes is of interest to both mathematicians and statisticians. In this thesis, we review the literature on the first passage time problem for both one-dimensional and two-dimensional diffusion processes. Then we investigate the statistical inference problem about unknown parameters of the Cox-Ingersoll-Ross model based on discretely observed first passage times. We are able to determine the identifiable parameter set, discuss the tail property of the density function in a neighborhood of the true parameter, and propose a conditional version of maximum likelihood estimation. We also list future work, including extensions of this problem to a general one-dimensional time homogeneous diffusion process, and to some special two-dimensional diffusion processes.

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1.0 INTRODUCTION

Stochastic modeling is a well-developed approach that has applications in many areas. A very important example is diffusion processes, which have become a prominent tool for modeling continuous-time evolution of phenomena not only in the natural sciences-physics, neuroscience, epidemiology-but also finance and economics. In 1900 L. Bachelier studied the Brownian motion model for stock markets. Later, Calvin and Stevens (Calvin and Stevens, 1965) proposed diffusion models for neural activity; Black and Scholes (Black and Scholes, 1974) modeled stock prices using diffusions, and derived a formula that plays an important role in pricing certain financial instruments. The Cox-Ingersoll-Ross (CIR) model is often used to describe the evolution of interest rates.

In recent years, both mathematicians and statisticians are becoming more and more interested in inference for certain parameters of diffusion processes. There are mainly two types of problems: one deals with the statistical inference based on discretely-observed diffusions, the other is based on their first passage times. In both cases because of the intractability of the densities, difficulty arises when we try to apply the classical maximum likelihood estimation (MLE) method. For the parametric inference of discretely observed diffusions, Ait-Sahalia (Aït-Sahalia, 2002) proposed an approach that involves the expansion of the transition density in a Gram-Chalier series. It works for one-dimensional diffusion processes with stochastic differential equation (SDE):

$$dX(t) = \mu(X(t), \theta)dt + \sigma(X(t), \theta)dB(t)$$

under weak regularity conditions. Here, B is Brownian motion, μ is the drift, σ is the standard deviation (σ^2 is called the diffusion coefficient), and X is the process of interest. Poulsen (Poulsen and Poulsen, 1999) started from Kolmogorov's forward equation and developed numerical solutions to access the likelihood function. Several other methods such as the simulated MLE by Pedersen

(Pedersen, 1995) and local linearization by Shoji and Ozaki (Shoji and Ozaki, 1998) are also well-studied. Those methods are based on approximations to the likelihood functions. Moreover, for the simulation approach, Beskos et al. (Beskos et al., 2005), (Beskos et al., 2006a), (Beskos et al., 2006b) introduced an exact algorithm which enables exact simulation of diffusion paths without any discretization of time.

Besides the statistical inference problem of observed diffusion processes, the study of the first passage time is also a huge topic in both mathematics and statistics, which has many important applications. In short, instead of the diffusion itself, it investigates the first passage time to a certain level of boundary of some diffusion processes. For example, in neuroscience, Gerstein and Mandelbrot (Gerstein and Mandelbrot, 1964) proposed the earliest integrate-and-fire diffusion model for single neuron activity. They approximated it by a Brownian motion with constant drift and diffusion coefficients,

$$dX_t = \mu dt + \sigma dB_t,$$

with X hitting a constant boundary corresponding to the firing of the neuron; the first passage time probability density density function (pdf) is the well known inverse Gaussian. Later, Stein (Stein, 1965) introduced models that also took into account of the membrane potential decay through leakage to derive the Ornstien-Uhlenbeck (OU) process that passes a constant boundary. The SDE is

$$dX_t = \left(\mu - \frac{X_t}{\tau}\right) dt + \sigma dB_t.$$

Unlike Brownian motion, the pdf of the first passage time through a constant boundary for an OU process is intractable. Ricciardi and Sato (Ricciardi and Sato, 1988) studied the first passage time density and moments of the OU process. They provided its Laplace transform that involves a ratio of parabolic cylinder functions, along with an iterative formula to calculate n^{th} moment. Meanwhile, they showed the tail probability of the pdf is exponentially decaying. Mullowney and Iyengar () proposed a method for statistical inference based on constant boundary crossing time of the OU process.

There is also some literature on non-constant boundary crossing times. Instead of a constant boundary, the boundary is a known time-dependent function $b(t)$. The most common case is to investigate the first passage time of a Brownian motion to a moving boundary. Uchiyama (Uchiyama, 1980) obtained the result that under some integral test being satisfied, the asymptotic probability of the Brownian motion to a moving boundary is the same as it is to a constant boundary. Valov (Valov, 2009) investigated the behavior of non-constant boundary crossing time of the Brownian motion using integral equation methods. Furthermore, based on Girsanov's theorem and Dambis-Dubins-Schwarz time change theorem, many processes can be transformed into a Brownian motion. The constant boundary crossing problem for certain general diffusion processes can also be converted into a moving boundary crossing problem for a Brownian motion.

Most of literature on the boundary crossing problem develops methods to obtain the pdfs of the first passage time given the known diffusion dynamics parameters μ and σ. Only a few of them deal with inference, such as estimation of unknown parameters, confidence intervals, and asymptotic efficiency. My interest and part of the goal of this thesis is to investigate statistical inference method for a time homogeneous diffusion process with the SDE:

$$dX_t = \mu(X_t)dt + \sigma(X_t)dB_t.$$

Meanwhile, much of the research above is focused on one-dimensional diffusion processes. For higher dimensions (even two !) the statistical modeling and mathematical problems are much more difficult. Higher dimensional diffusions are increasingly becoming important because of advances in many fields: they can be used to study networks of neurons or collections of financial instruments like stocks and options. However, due to the mathematical complexity, there is only a relatively limited literature. Iyengar (Iyengar, 1985) discussed the first passage time for a driftless correlated two-dimensional Brownian motion to constant boundaries. Kou and Zhong (Kou et al., 2016) pushed it further by obtaining the Laplace transform and its numerical inversion for the case with a constant drift. Moreover, Sacerdote et al. (Sacerdote et al., 2012) proposed a method to analyze the first passage time to a constant boundary of a general bivariate time-homogeneous diffusion process using Volterra-Fredholm integral equations. One goal of this thesis is to study the statistical inference problem for certain bivariate diffusion processes.

This thesis is organized thus: in Chapter 2, a literature review is presented for the boundary crossing problem of both one-dimensional diffusion process and two-dimensional process. In Chapter 3, we start with the theory of making statistical inference for the one-dimensional CIR model based on its first passage time. A general structure including the OU, the reflected Ornstien-Uhlenbeck (ROU) and the CIR will be discussed in Chapter 4. In Chapter 5, the bivariate problem is addressed. Work mentioned in Chapter 4 and Chapter 5 are mainly left for the future. The proofs of all lemmas and theorems can be found in the appendix.

2.0 LITERATURE REVIEW

2.1 FIRST PASSAGE TIME PROBLEM FOR A ONE-DIMENSIONAL DIFFUSION PROCESS

In this section, we review some literature on the boundary crossing problem for one-dimensional diffusion processes. We introduce here the first passage time to a constant boundary for the Brownian motion, the OU process, the CIR process and a general Markov process, along with the methods used to analyze them.

2.1.1 Brownian motion

The Brownian motion is named after the botanist Robert Brown. Starting with the work of Wiener () and Bachelier (), its properties have been well studied and can be found in any classical stochastic process textbook. The first passage time of a standard Brownian motion B_t to a constant boundary $a > 0$ is known to have an inverse Gaussian pdf:

$$f(t) = \frac{a}{\sqrt{2\pi}} t^{-\frac{3}{2}} e^{-\frac{a^2}{2t}}$$

For a Brownian motion with a constant drift μ, by Girsanov's theorem, we know the pdf f_μ for the first passage time of a has the form:

$$f_\mu(t) = \frac{a}{\sqrt{2\pi}} t^{-\frac{3}{2}} e^{-\frac{(a-\mu t)^2}{2t}}$$

Because those pdfs are simple and form an exponential family, given discretely observed first passage times t_1, t_2,.., t_n, classical theory of the MLE method can be applied to obtain efficient estimates of the unknown parameters μ and a. We have asymptotic normality and efficiency for the MLE, thus we can do standard statistical inference using the information matrix.

2.1.2 Ornstien-Uhlenbeck process

Instead of a constant drift, if the drift is mean-reverting and proportional to the diffusion itself, we have an OU process, with the SDE:

$$dX_t = -\frac{X_t}{\tau}dt + \sigma dB_t, \text{ with } \tau > 0, \sigma > 0.$$

Define the first passage time through a boundary x for X_t starting at x_0 to be $T = \inf\{t : X_t = x\}$, so that T is a stopping time. However, unlike the Brownian motion, the pdf of T is complicated. In (Alili et al., 2005), several approaches have been discussed, including the series representation, the integral representation and the Bessel Bridge representation. In fact, we do not a have direct access to its pdf $f(t)$, but we can work with its Laplace transform: $\hat{f}(s) = E(e^{-sT}) = \int_0^\infty e^{-st}f(t)dt$. The explicit form of $\hat{f}(s)$ is given in (Ricciardi and Sato, 1988):

$$\hat{f}(s) = \begin{cases} e^{(\frac{x_0^2-x^2}{2\sigma^2\tau})}\dfrac{D_{-s\tau}(-x_0\sqrt{2/\sigma^2\tau})}{D_{-s\tau}(-x\sqrt{2/\sigma^2\tau})} & \text{if } x_0 < x \\ e^{(\frac{x_0^2-x^2}{2\sigma^2\tau})}\dfrac{D_{-s\tau}(-x_0\sqrt{2/\sigma^2\tau})}{D_{-s\tau}(x\sqrt{2/\sigma^2\tau})} & \text{if } 0 < x < x_0 \end{cases} \tag{2.1.1}$$

where $D_\lambda(z)$ is the parabolic cylinder function (Lebedev).

For the $0 < x_0 < x$ and with an additional μ in the drift, Iyengar (Iyengar and Mullowney, 2007) used the expression

$$\hat{f}(s) = \frac{M(\theta_1, s\theta_3)}{M(\theta_2, s\theta_3)} \tag{2.1.2}$$

where $(\theta_1, \theta_2, \theta_3) = (\frac{x_0-\mu\tau}{\sigma\sqrt{\tau}}, \frac{x-\mu\tau}{\sigma\sqrt{\tau}}, \tau)$ and $M(z, v)$ is the Hermite function. He also proved that merely given first passage time observations, the identifiable parameters are $(\theta_1, \theta_2, \theta_3)$ rather than the original (μ, σ, x_0, x). Using the asymptotic expansions of Hermite functions, Iyengar was able to show the validity of the Bromwich integral to invert $\hat{f}$ and its first three partial derivatives with respect to parameter $(\theta_1, \theta_2, \theta_3)$. By checking the classical conditions for the MLE in (Lehmann and Casella, 2006), Iyengar obtained a result of asymptotic normality and efficiency for the MLE. In (Mullowney and Iyengar, 2008), Mullowney and Iyengar applied the OU model to a real data, with the MLE in (Iyengar and Mullowney, 2007); they were able to construct asymptotic confidence intervals for the unknown identifiable parameters.

2.1.3 Cox-Ingersoll-Ross process

The CIR model is commonly used in mathematical finance and neuroscience, and it is also known as the Feller process. The SDE associated with the CIR model is:

$$dY_t = [-\alpha Y_t + \beta]dt + k\sqrt{Y_t}dB_t, \quad \alpha, \beta, k > 0$$

Jaeschke and Yor (Göing-Jaeschke et al., 2003) carefully studied the first passage time that Bessel processes and radial OU processes hit a constant boundary. Since CIR process has the same dynamic with radial OU process, they are closed related. The authors also gave an explicit expression for the Laplace transform of the first passage time through a constant boundary for the CIR process. They used a martingale technique to construct a partial differential equation (PDE), the solution to which is the Laplace transform. The Laplace transform can be also obtained through a PDE arising from the (forward) Fokker-Planck equation (Masoliver and Perelló, 2012).

If we let T_{y_c} be the first time of Y_t starting from y crosses y_c, then its Laplace transform is:

$$Ee^{-sT_{y_c}} = \begin{cases} \dfrac{F(\frac{s}{\alpha}, \frac{2\beta}{k^2}, \frac{2\alpha y}{k^2})}{F(\frac{s}{\alpha}, \frac{2\beta}{k^2}, \frac{2\alpha y_c}{k^2})} & \text{if } 0 < y < y_c \\ \dfrac{U(\frac{s}{\alpha}, \frac{2\beta}{k^2}, \frac{2\alpha y}{k^2})}{U(\frac{s}{\alpha}, \frac{2\beta}{k^2}, \frac{2\alpha y_c}{k^2})} & \text{if } 0 < y_c < y \end{cases}$$

where F and U are confluent hypergeometric functions of the first and the second kind (Abramowitz and Stegun, 1965). They also carefully studied the behavior of the CIR process hitting the origin and a large threshold, along with the mean of the hitting time and the tail behavior of its pdf.

Moreover, the Laplace transform is not the only tool with which we can access the pdf of the boundary crossing time of the OU and the CIR process. Linetsky (Linetsky, 2004) proved that under some conditions, the pdf of the first passage time to y for a diffusion process starting at x can be written as:

$$f_{x\to y}(t) = \sum_{n=1}^{\infty} c_n \lambda_n e^{-\lambda_n t} \tag{2.1.3}$$

where $\{\lambda_n\}$, $0 < \lambda_1 < \lambda_2 < ..., < \lambda_n \to \infty$ as $n \to \infty$. For all $t_0 > 0$, the series converges uniformly on $[t_0, \infty]$. Equation (2.1.3) works for both the OU process and the CIR process. The author also did some numerical studies of this expansion.

Although the CIR model is commonly used and there are some different methods to estimate the pdf of the boundary crossing time, only a little literature considers statistical inference for these models. Thus, given only the observed stopping times: t_1, t_2, ..., t_n, we do not know what are the identifiable functions of the parameters $(y, y_c, \alpha, \beta, k)$, nor do we know how to construct estimates and classical confidence intervals for them. In Chapter 3, We investigate these two issues carefully and prove a theorem about the MLE to do statistical inference of the boundary crossing time for the CIR model.

2.1.4 General one-dimensional process

We have seen that even for the simple processes such as the OU and the CIR model, the density function of the first passage time to a constant boundary can be analytically intractable. Meanwhile, from (Siegert, 1951) and (Darling and Siegert, 1953) we know that the accessibility of the Laplace transform of the first passage time T relies heavily on the resolvability of some ordinary or partial differential equations (ODEs or PDEs). It can be very difficult to perform statistical inference for general diffusion processes. The main challenge is to obtain the density functions; furthermore the construction of confidence intervals for the unknown parameters is even more difficult.

There is some literature on estimating the densities of the first passage time for general diffusion processes. In 1987, Buonocore et al. (Buonocore et al., 1987) used a Volterra integral equation to access the first passage time density function of a time-homogeneous diffusion process through a time-dependent boundary. They proposed a numerical procedure to obtain a solution and showed its convergence. They also discussed several cases when closed form results exist, such as the Brownian motion to a boundary linear in time, and the OU process through a hyperbolic boundary. Gutierrez, et al. (Gutiérrez et al., 1997) studied the first passage time density functions through a time-dependent boundary for time-non-homogeneous diffusion processes. They used a Volterra integral equation of the second kind. Later Buonocore et al. (Buonocore et al., 2011), (Buonocore et al., 2015) revisited the problem by analyzing the first passage time of a Gaussian diffusion processes and applied it to a leaky integrate-and-fire (LIF) neuronal model. We now outline their methods briefly. Consider a real-valued Gaussian diffusion processes $X(t)$ with transition pdf $f(x, t|y, \tau)$ and transition cumulative distribution function (cdf) $F(x, t|y, \tau)$ satisfying the Fokker-Planck equation:

$$\frac{\partial f(x,t|,y,\tau)}{\partial t} = -\frac{\partial}{\partial x}[A_1(x,t)f(x,t|y,\tau)] + \frac{1}{2}\frac{\partial^2}{\partial x^2}[A_2(t)f(x,t|y,\tau)] \quad (t_0 \le \tau \le t) \tag{2.1.4}$$

with initial condition

$$\lim_{t \downarrow t_0} f(x,t|x_0,t_0) = \delta(x - x_0)$$

where infinitesimal moments $A_1(x,t) = a(t)x + b(t)$ and $A_2(t) = \sigma^2(t)$ $(t > t_0)$, along with mean and autocovariance structure $E[X(t)] = m(t)$ in $C^1[t_0,\infty)$ and $E\{[X(\tau) - m(\tau)][X(t) - m(t)]\} = c(\tau,t)$ in $C^1([t_0,\infty) \times [t_0,\infty))$.

With the Markov property, there are two C^1 functions such that: (i) for $t_0 < \tau \le t$, $c(\tau,t) = h_1(\tau)h_2(t)$; (ii) $Var(X(t)) = h_1(t)h_2(t) > 0$, (iii) $r(t) = \frac{h_1(t)}{h_2(t)}$ is increasing in t. By Gaussian property, the conditional mean and conditional variance can be written thus:

$$M(t|y,\tau) = E[X(t)|X(\tau) = y] = m(t) + \frac{h_2(t)}{h_2(\tau)}[y - m(\tau)] \quad (t_0 < \tau < t)$$

$$D^2(t|\tau) = Var[X(t)|X(\tau) = y] = h_2(t)[h_1(t) - \frac{h_2(t)}{h_2(\tau)}h_1(\tau)] \quad (t_0 < \tau < t)$$

Combining with Equation (2.1.4) and the form of A_1, A_2, we can obtain:

$$a(t) = \frac{h_2'(t)}{h_2(t)}, \quad b(t) = m'(t) - m(t)\frac{h_2'(t)}{h_2(t)}, \quad \sigma^2(t) = h_2^2(t)r'(t).$$

Let $A(t) = \int_0^t a(s)ds$ and assume some integrability conditions, to get:

$$m(t) = \left[x_0 + \int_{t_0}^t b(s)e^{-A(s)}ds\right]e^{A(t)}$$

$$c(\tau,t) = e^{A(t)}e^{A(\tau)}\int_{t_0}^\tau \sigma^2(s)e^{-2A(s)}ds$$

Therefore

$$h_1(t) = \frac{e^{A(t)}}{\sigma(t_0)}\sigma^2(s)e^{-2A(s)}ds$$

$$h_2(t) = \sigma(t_0)e^{A(t)}$$

Then Gauss-Markov process $X(t)$ has a Brownian motion representation:

$$X(t) = m(t) + h_2(t)B[r(t)] \tag{2.1.5}$$

where $B(t)$ is the standard Brownian motion. For the boundary crossing problem of $X(t)$ through $S(t) \in C^1([t_0, \infty)$, set

$$T_{x_0} = \inf_{t \geq t_0} \{X(t) > S(t)\}, \quad X(t_0) = x_0 < S(t_0)$$

The first method to study the boundary crossing problem of a Brownian motion to some time-dependent boundary is based on the representation (2.1.5). The second method is to construct an integral equation system by the Markov property, let:

$$g[S(t), t|x_0, t_0] = \frac{\partial}{\partial t} P(T_{x_0} \leq t) \quad \text{and} \quad \phi(x, t|y, \tau) = \frac{\partial (F(x, t|y, \tau)}{\partial t}.$$

Conditioning on the first passage time T_{x_0} with $s(t_0) > x_0$, we get:

$$1 - F(S(t), t|x_0, t_0) = \int_{t_0}^{t} g[S(\tau), \tau|x_0, t_0] \times \{1 - F[S(t), t|S(\tau), \tau]\} d\tau \tag{2.1.6}$$

Differentiate Equation (2.1.6) with respect to t:

$$\begin{aligned} & - S'(t) \frac{\partial F(x, t|x_0, t_0)}{\partial x}|_{x=S(t)} - \phi(S(t), t|x_0, t_0) \\ =& g[S(t), t|x_0, t_0] - g[S(t), t|x_0, t_0] F[S(t), t|S(t^-, t^-)] \\ & - \int_{t_0}^{t} g[S(\tau), \tau|x_0, t_0] \{S'(t) \frac{\partial F(x, t|x_\tau, \tau)}{\partial x}|_{x=s(t)} + \phi(S(t), t|S(\tau), \tau)\} d\tau \end{aligned} \tag{2.1.7}$$

With

$$\int_{t_0}^{t} g[S(\tau), \tau|x_0, t_0] \{S'(t) \frac{\partial F(x, t|x_\tau, \tau)}{\partial x}|_{x=s(t)}\} d\tau = S'(t) \frac{\partial F(x, t|x_0, t_0)}{\partial x}|_{x=s(t)}$$

and (Fortet, 1943)

$$F(S(t), t|S(t^-, t^-) = \frac{1}{2}$$

we obtain

$$g[S(t), t|x_0, t_0] = 2\phi(S(t), t|x_0, t_0) - 2 \int_{t_0}^{t} g[S(\tau), \tau|x_0, t_0] \phi(S(t), t|S(\tau), \tau) d\tau \tag{2.1.8}$$

For $k(t)$, $l(t)$ continuous, $\tau < t$, define

$$\psi[S(t), t|y, \tau] = \phi(S(t), t|y, \tau) + k(t) \frac{\partial F(x, t|y, \tau)}{\partial x}|_{x=S(t)} + l(t)[1 - F(S(t), t|y, \tau)]. \tag{2.1.9}$$

Then a Volterra integral equation of the second kind appears:

$$g[S(t), t|x_0, t_0] = -2\psi[S(t), t|x_0, t_0] + 2\int_{t_0}^{t} g[S(\tau), \tau|x_0, t_0]\psi[S(t), t|S(\tau), \tau]d\tau; \qquad (2.1.10)$$

for cases that $l(t) = 0$, a simpler form of $\psi[S(t), t|y, \tau]$ can be derived:

$$\psi[S(t), t|y, \tau] = \{S'(t) - A_1[S(t), t] - A_2(t)\frac{S(t) - M(t|y, \tau)}{2D^2(t|\tau)} + k(t)\} \times f[S(t), t|y, \tau]. \qquad (2.1.11)$$

The authors investigated several cases when Equation (2.1.8) and Equation (2.1.11) has a closed form solution. The idea is to set kernel ψ to be 0. From the above analysis, we can see that Equation (2.1.10) can be applied to general diffusion processes, while the only condition is the Markov property. Equation (2.1.11) is derived by the special Gaussian structure.

Integral equations can also arise in some other ways when dealing with the first passage time problem. In Valov's thesis (Valov, 2009), the first passage time of the Brownian motion to a time-dependent boundary is revisited. By constructing a series of martingales and using the optional sampling theorem, he was able to show a new system of integral equations. Other literature (Fu and Wu, 2010), (Ji and Shao, 2015) estimated the first passage time densities by approximating the general processes with Markov chains and linearizing the time-dependent boundary.

However, most of the literature is focused on estimating the densities of the first passage time given the characteristics of the diffusion process. Little of it dealt with the statistical inference. Given only the observed first passage times and general dynamic structure

$$dX(t) = \mu(X(t), \theta)dX(t) + \sigma(X(t), \theta)dB(t),$$

where μ and σ are known smooth functions, even point estimation of θ can be very difficult, let alone dealing with statistical inference. The purpose of this thesis is to start with statistical inference for some special processes, such as OU, ROU and CIR models, and seek a unified a way to deal with more general processes.

2.2 FIRST PASSAGE TIME PROBLEM FOR A TWO-DIMENSIONAL DIFFUSION PROCESS

As we mentioned in the introduction, when you increase the dimensionality, the first passage time problem for diffusion processes becomes much more complicated. In this section, we list some known results for the first passage time problem of the two-dimensional Brownian motion and other general diffusions.

2.2.1 Two-dimensional Brownian motion

Given $X = (X_1, X_2)$ a two-dimensional correlated Brownian motion starting at x_0, let τ_i be the first passage time of X_i to the constant boundary 0. The case without drift was investigated by Iyengar (Iyengar, 1985), and Metzler (Metzler, 2010) restudied the problem. Here we outline some of their methods and results.

Consider a two-dimensional Brownian motion X_t starting at x_0:

$$dX(t) = \sigma dB(t), \text{ where } \sigma = \begin{bmatrix} \sigma_1\sqrt{1-\rho^2} & \sigma_1\rho \\ 0 & \sigma_2 \end{bmatrix}$$

By linear transformation $Z(t) = \sigma^{-1}X(t)$, we know $Z(t)$ is an uncorrelated two-dimensional Brownian motion. The horizontal axis is invariant under transformation, while the vertical axis is mapped to $z_1 = -\frac{\rho}{\sqrt{1-\rho^2}}z_2$. Then τ_2 is redefined as the first passage time of $Z(t)$ to horizontal axis, τ_1 is redefined as the first passage time of $Z(t)$ to the line $z_2 = z_1 \tan\alpha$:

$$\alpha = \begin{cases} \pi + \tan^{-1}(-\frac{\sqrt{1-\rho^2}}{\rho}) & \text{if } x_0 < x \\ \frac{\pi}{2} & \text{if } \rho = 0 \\ \tan^{-1}(-\frac{\sqrt{1-\rho^2}}{\rho}) & \text{if } \rho < 0 \end{cases}$$

$Z(t)$ starts at $z_0 = (r_0 \cos\theta_0, \sin\theta_0)$, where

$$r_0 = \sqrt{\frac{a_1^2 + a_2^2 - 2\rho a_1 a_2}{1-\rho^2}}$$

$$\theta_0 = \begin{cases} \pi + \tan^{-1}(-\frac{a_1\sqrt{1-\rho^2}}{a_1 - a_2\rho}) & \text{if } a_1 < \rho a_2 \\ \frac{\pi}{2} & \text{if } a_1 = \rho a_2 \\ \tan^{-1}(-\frac{a_1\sqrt{1-\rho^2}}{a_1 - a_2\rho}) & \text{if } a_1 > \rho a_2 \end{cases}$$

The problem now is recast as the boundary crossing problem of $Z(t)$ through $\partial C_\alpha = \{(r\cos\theta, r\sin\theta) : r > 0, 0 < \theta < \alpha\}$ with C_α being an infinite wedge.

Let $\tau = \min(\tau_1, \tau_2)$, Iyengar derived:

$$P(\tau > t, Z(t) \in dz) = \frac{2r}{t\alpha} e^{-(r^2+r_0^2)/2t} \sum_{n=1}^{\infty} \sin\frac{n\pi\theta}{\alpha} \sin\frac{n\pi\theta_0}{\alpha} I_{n\pi/\alpha}(\frac{rr_0}{t}) \quad (2.2.1)$$

where $I_v(x)$ is the modified Bessel function of the first kind of order v. Integrating over the wedge, one can obtain:

$$P(\tau > t) = \frac{2r_0}{\sqrt{2\pi t}} e^{-r_0^2/4t} \sum_{nodd} \frac{1}{n} \sin\frac{n\pi\theta_0}{\alpha} [I_{(v_n-1)/2}(r_0^2/4t) + I_{(v_n+1)/2}(r_0^2/4t)] \quad (2.2.2)$$

On the other hand, the joint density of $(\tau, Z(\tau))$ is also known to be

$$P(\tau \in dt, Z(\tau) \in dz) = \frac{1}{2}\frac{\partial}{\partial n} P(Z(t) \in dz, \tau > t) \quad (2.2.3)$$

where the partial derivative denotes derivative in the direction of the inward normal to the boundary αC_α. Metzler (Metzler, 2010) also had expressions for the density function of $Z(\tau)$ and joint density function (τ_1, τ_2):

$$P(R(\tau) \in dr, \Theta(\tau) = 0) = \frac{dr}{\alpha_2 r_0} \frac{(r/r_0)^{(\pi/\alpha)-1} \sin(\pi\theta_0/\alpha)}{\sin^2(\pi\theta_0/\alpha) + [(r/r_0)^{\pi/\alpha} - \cos(\pi\theta_0/\alpha)]^2} \quad (2.2.4)$$

$$P(R(\tau) \in dr, \Theta(\tau) = \alpha) = \frac{dr}{\alpha_2 r_0} \frac{(r/r_0)^{(\pi/\alpha)-1} \sin(\pi\theta_0/\alpha)}{\sin^2(\pi\theta_0/\alpha) + [(r/r_0)^{\pi/\alpha} + \cos(\pi\theta_0/\alpha)]^2}. \quad (2.2.5)$$

For $s < t$,

$$\begin{aligned} P(\tau_1 \in ds, \tau_2 \in dt) =& \frac{ds\ dt\ \pi \sin\alpha}{2\alpha^2\sqrt{s(t - s\cos^2\alpha)}(t-s)} \exp\left(-\frac{r_0^2}{2s}\frac{t - s\cos 2\alpha}{(t-s) + (t - s\cos 2\alpha)}\right) \\ & \times \sum_{n=1}^{\infty} n \sin(\frac{n\pi(\alpha - \theta_0)}{\alpha}) I_{n\pi/2\alpha}\left(\frac{r_0^2}{2s}\frac{t - s\cos 2\alpha}{(t-s) + (t - s\cos 2\alpha)}\right) \end{aligned} \quad (2.2.6)$$

For $s > t$:

$$P(\tau_1 \in ds, \tau_2 \in dt) = \frac{ds\ dt\ \pi \sin\alpha}{2\alpha^2\sqrt{t(s - t\cos^2\alpha)}(s-t)} \exp\left(-\frac{r_0^2}{2t}\frac{s - t\cos 2\alpha}{(s-t)+(s-t\cos 2\alpha)}\right) \times \sum_{n=1}^{\infty} n \sin(\frac{n\pi\theta_0}{\alpha}) I_{n\pi/2\alpha}\left(\frac{r_0^2}{2t}\frac{s - t\cos 2\alpha}{(s-t)+(s-t\cos 2\alpha)}\right) \tag{2.2.7}$$

Neither Iyengar nor Metzler gave an explicit expression for those densities when some constant drift is present, although Metzler approached the problem by Monte Carlo simulation. Kou and Zhong (Kou et al., 2016) investigated the case with drift. Adding constant drift μ_i to X_i, with a martingale argument, they showed the solution to the following partial differential equation is $u(x_0) = E[e^{-p_1\tau_1 - p_2\tau_2 - v|\tau_2-\tau_1|}]$:

$$\frac{1}{2}\sigma_1^2\frac{\partial_2^2 u}{\partial x_1^2} + \rho\sigma_1\sigma_2\frac{\partial^2 u}{\partial x_1 \partial x_2} + \frac{1}{2}\sigma_2^2\frac{\partial_2^2 u}{\partial x_2^2} + \mu_1\frac{\partial u}{\partial x_1} + \mu_2\frac{\partial u}{\partial x_2} = cu \tag{2.2.8}$$

with boundary conditions:

$$\begin{cases} u(x_1, x_2)|_{x_1=0} = \exp(-D_2 x_2) \\ u(x_1, x_2)|_{x_2=0} = \exp(-D_1 x_1) \end{cases}$$

where $p_1 = \frac{1}{4}\sigma_1^2 D_1^2 - \frac{1}{2}\mu_1 D_1 - \frac{1}{4}\sigma_2^2 D_2^2 + \frac{1}{2}\mu_2 D_2 + \frac{1}{2}c$, $p_2 = -p_1 + c$, and $v = \frac{1}{4}\sigma_1^2 D_1^2 - \frac{1}{2}\mu_1 D_1 + \frac{1}{4}\sigma_2^2 D_2^2 - \frac{1}{2}\mu_2 D_2 + \frac{1}{2}c$.

D_1 and D_2 was derived from the Laplace transform of the first passage time for the one-dimensional Brownian motion with a drift:

$$D_i = \frac{\sqrt{\mu_i^2 + 2(p_j + v)\sigma_i^2} + \mu_i}{\sigma_i^2}$$

They proved that the solution to (2.2.8) is unique:

$$u_1(x_1, x_2) = e^{-(\gamma_1\cos\theta_0 + \gamma_2\sin\theta_0)r} \sum_{n=1}^{\infty} \sqrt{\frac{2}{\alpha}} \sin(v_n\theta_0) u_{1,n}(r) + e^{-D_1x_1 - D_2x_2} \tag{2.2.9}$$

where $v_n = \frac{n\pi}{\alpha}$, $\gamma_1 = \frac{\mu_1/\sigma_1 - \rho\mu_2/\sigma_2}{1-\rho^2}$, $\gamma_2 = \mu_2/\sigma_2$, $A = \sigma_1^2 D_1^2 + 2\rho\sigma_1\sigma_2 D_1 D_2 + \sigma_2^2 D_2^2 - 2\mu_1 D_1 - 2\mu_2 D_2 - 2c$, $G(\theta) = -\gamma_1\cos\theta - \gamma_2\sin\theta + D_1\sigma_1\sin(\alpha-\theta) + D_2\sigma_2\sin\theta$, $a = \sqrt{2c + \gamma_1^2 + \gamma_2^2}$, and

$$u_{1,n}(r) = \frac{1}{2}A\int_0^\alpha \sqrt{\frac{2}{\alpha}}\sin(v_n\eta)[K_{v_n}(ar)\int_0^r \exp(-G(\eta)l) l I_{v_n}(al)dl + I_{v_n}(ar)\int_r^\infty \exp(-G(\eta l) l K_{v_n}(al)dl]d\eta$$

To compute the density function of τ or joint density of τ_1 and τ_2, one still needs to numerically invert Equation (2.2.9).

From all the existing results above, one can observe that even for the case without a drift, the density function of the first passage time of a two-dimensional Brownian motion through an infinite wedge involves infinite sums and derivatives of modified Bessel function, and for the case with a drift we need to invert Equation (2.2.9). The statistical inference problem of the unknown parameters ρ, σ_1, σ_2, μ_1 and μ_2 are therefore very challenging.

2.2.2 General bivariate diffusion process

For the first passage time problem of a general bivariate diffusion process, there is relatively limited literature. Sacerdote et al. (Sacerdote et al., 2012) proposed a method using the strong Markov property to get a system of Volterra-Fredholm integral equations. Consider a two-dimensional time homogeneous diffusion process $X(t) = (X_1(t), X_2(t), t > t_0)$, with SDE

$$dX_t = \mu(X(t))dt + \Sigma^{\frac{1}{2}}(X(t))dB_t, \quad X(t_0) = (x_{01}, x_{02})$$

where B_t is a standard two-dimensional Brownian motion, μ and Σ are measurable functions in $\mathcal{R}^2$ that makes the SDE have a solution in the strong sense and with strong Markov property.

Let $\tau_i = \inf\{t > t_0, X_i(t) \geq B_i\}$ be the first time that X_i goes beyond B_i with $B_i > x_{0i}$. $\tau = \min(\tau_1, \tau_2)$. Define a key joint density function of $X_i(\tau_j)$ with $\tau_i > \tau_j$:

$$f^a_{X_i,\tau_j}(x_i, t|y, s)dx_i dt = P(X_i \in dx_i, \tau_j \in dt, \tau_i > \tau_j | X(s) = y) \quad x < B_i, s < t \tag{2.2.10}$$

Conditioning on the first passage time τ and the position of the other element that has not passed the boundary (note that passing B_i and B_j simultaneously has probability zero) :

$$\begin{aligned}P(X(t)\geq x) =& P(X(t)\geq x,\tau_1<\tau_2)+P(X(t)\geq x,\tau_2>\tau_3)\\
=&\int_{t_0}^{t}\int_{-\infty}^{B_2}P(X(t)\geq x|X_{2s}(s)\in dx_2,\tau_1\in ds,\tau_1<\tau_2)\\
&\times P(X_2\in dx_2,\tau_1\in ds,\tau_1<\tau_2|X_{t_0}=x_0)\\
&+\int_{t_0}^{t}\int_{-\infty}^{B_1}P(X(t)\geq x|X_1\in dx_1,\tau_2\in ds,\tau_2<\tau_1)\\
&\times P(X_1\in dx_1,\tau_2\in ds,\tau_2<\tau_1|X_{t_0}=x_0)\\
&\quad\text{by strong Markov property}\\
=&\int_{t_0}^{t}\int_{-\infty}^{B_2}P(X(t)\geq x|X_2(s)\in dx_2,X_1\in dB_1)f^a_{X_1,\tau_2}(x_1,s|x_0,t_0)\\
&+\int_{t_0}^{t}\int_{-\infty}^{B_1}P(X(t)\geq x|X_1(s)\in dx_1,X_2\in dB_2)f^a_{X_2,\tau_1}(x_2,s|x_0,t_0)\end{aligned}\tag{2.2.11}$$

where $x=(x_1,x_2)$ and $x_1>B_1$, $x_2>B_2$.

Let $x_1\downarrow B_1$ and $x_2\downarrow B_2$ respectively, and define $\hat{F}(x,t|y,s)=P(X(t)\geq x|X(s)=y)$ to get a system of Vottera-Fredholm integral equations:

$$\begin{aligned}\hat{F}((x_1,B_2),t|x_0,t_0)=&\int_{t_0}^{t}\int_{-\infty}^{B_2}\hat{F}((x_1,B_2,t)|(B_1,y)f^a_{X_2,\tau_1}(y,s|x_0,t_0)dyds\\
&+\int_{t_0}^{t}\int_{-\infty}^{B_1}\hat{F}((x_1,B_2,t)|(y,B_2)f^a_{X_1,\tau_2}(y,s|x_0,t_0)dyds\end{aligned}\tag{2.2.12}$$

$$\begin{aligned}\hat{F}((B_1,x_2),t|x_0,t_0)=&\int_{t_0}^{t}\int_{-\infty}^{B_2}\hat{F}((B_1,x_2,t)|(B_1,y)f^a_{X_2,\tau_1}(y,s|x_0,t_0)dyds\\
&+\int_{t_0}^{t}\int_{-\infty}^{B_1}\hat{F}((B_1,x_2,t)|(y,B_2)f^a_{X_1,\tau_2}(y,s|x_0,t_0)dyds\end{aligned}\tag{2.2.13}$$

When the process is Gaussian, the form of $\hat{F}$ is available, as it is associated with a conditional distribution of a bivariate normal. The authors provided a numerical method to solve the Equation (2.2.12) and (2.2.13) and proved its convergence. With an estimate of $f^a_{X_i,\tau_j}(x_i,t|y,s)$, other joint density functions can be approached by integration. Meanwhile, they applied this method to the two-dimensional Brownian motion and the OU processes and did some error analysis.

From the review above, we can see that compared with the one dimensional case, even for the simplest two-dimensional Brownian motion, it is difficult to analyze its first passage time through constant boundaries. I have found no literature dealing with the statistical inference for this problem, and this is exactly the reason why we plan to do some research on problems of this type from a statistical viewpoint in Chapter 5.

3.0 STATISTICAL INFERENCE BASED ON THE FIRST PASSAGE TIME OF THE CIR PROCESS

In this chapter, we focus on the problem of doing inference on the first passage time of the CIR model through a constant boundary, especifically estimating the parameters and providing confidence intervals. Because the pdfs are analytically intractable, we approach it by its Laplace transform, and verify that the Bromwich integral can be used to invert the Laplace transform. Then we study the identifiability of the parameters, the asymptotic property of the pdfs and prove a conditional version of the MLE for the identifiable parameters.

3.1 THE MODEL AND SOME NOTATION

A CIR process Y_t starting at y has the following SDE and initial condition:

$$dY_t = [-\alpha Y_t + \beta]dt + k\sqrt{Y_t}dB_t, \text{ with } Y_0 \quad = y \tag{3.1.1}$$

where $\alpha > 0$, $\beta > 0$, $k > 0$, and B_t is standard Brownian motion.
Consider the first passage time of Y_t through a constant boundary y_c:

$$T_{y_c} = \inf\{t : Y_t = y_c\}$$

Stated in Section 2.1.3, the Laplace transform of T_{y_c} is

$$Ee^{-sT_{y_c}} = \begin{cases} \dfrac{F(\frac{s}{\alpha}, \frac{2\beta}{k^2}, \frac{2\alpha y}{k^2})}{F(\frac{s}{\alpha}, \frac{2\beta}{k^2}, \frac{2\alpha y_c}{k^2})} & \text{if } 0 < y < y_c \\ \dfrac{U(\frac{s}{\alpha}, \frac{2\beta}{k^2}, \frac{2\alpha y}{k^2})}{U(\frac{s}{\alpha}, \frac{2\beta}{k^2}, \frac{2\alpha y_c}{k^2})} & \text{if } 0 < y_c < y \end{cases} \tag{3.1.2}$$

where F and U are the confluent hypergeometric functions of the first and the second kind. As we are going to access the pdf of the first passage time by inverting a ratio of either F or U using the Bromwich integral, knowing the asymptotic properties of $F(a, b, c)$ and $U(a, b, c)$ when $|a| \to \infty$ in the complex plane is necessary.

3.2 ASYMPTOTIC EXPANSION OF F AND U

Asymptotic expansions for certain special functions have long been of interest in mathematics: for $F(a, b, c)$ and $U(a, b, c)$, we can find much literature on the asymptotic expansion for $|c| \to \infty$. However, fewer papers are written about the case of a. When a is real, Slater (Slater, 1960) gave an asymptotic expansion using modified Bessel functions I and K as $a \to +\infty$. Temme (Temme, 2013) revisited the problem and revised Slater's result. But neither of them extended it to the complex case of a. Eventually in 2016, Volkmer (Volkmer et al., 2016) managed to work with the complex a by proving the following theorem in (Volkmer et al., 2016): *Suppose that $b \in \mathbb{C}$ is not 0 or a negative integer, $u = te^{i\theta}$ with $t > 0$, $\theta \in \mathcal{R}$, and $N \geq 1$, $R > 0$. Then*

$$\begin{aligned} &\frac{2^{1-b}u^{b-1}}{\Gamma(b)} e^{-\frac{1}{2}z^2} z^b F(\frac{1}{4}u^2 + \frac{1}{2}b, b, z^2) \\ &= zI_{b-1}(uz)(\sum_{s=0}^{N-1} \frac{A_s(z)}{u^{2s}} + g_1(u, z)) + \frac{z}{u} I_b(uz)(\sum_{s=0}^{N-1} \frac{B_s(z)}{u^{2s}} + zh_1(u, z)) \end{aligned} \tag{3.2.1}$$

where I_v is the modified Bessel function of the first kind of order v, $|g_1(u, z)| + |h_1(u, z)| \leq \frac{L_1}{t^{2N}}$ for $0 < |z| \leq R$, $t \geq t_1$. L_1, t_1 are positive constants independent of z and u (but possibly depending on b, θ, N, R). $A_s(z)$ and $B_s(z)$ are polynomials that can be obtained recursively. In fact, we will only use $A_0(z) = 1$ and $B_0(z) = \frac{1}{6}z^3$.

In the theorem above, we find that L_1 might depend on θ, while we would like an expansion that makes g_1 and h_1 be bounded by a constant free of θ. So we prove the following lemma:

Lemma 3.2.1. *With the same conditions required for (3.2.1), we have that for $\Re b > 0$:*

$$F(\frac{1}{4}u^2 + \frac{1}{2}b, b, z^2) = \frac{\Gamma(b)}{2^{1-b}u^{b-1}} e^{\frac{1}{2}z^2} z^{1-b}\Big(I_{b-1}(uz)(\sum_{s=0}^{N-1} \frac{\tilde{A}_s(e^{i\theta}z)}{t^{2s}} + g_{21}(t, e^{i\theta}z)) + \frac{1}{t} I_b(uz)(\sum_{s=0}^{N-1} \frac{\tilde{B}_s(e^{i\theta}z)}{t^{2s}} + e^{i\theta} z h_{21}(t, e^{i\theta}z))\Big) \tag{3.2.2}$$

where $u = te^{i\theta}$, $|g_{2i}| + |h_{2i}| \leq \frac{K_1}{t^{2N}}$ for $0 < |z| \leq R$, $t \geq t_1$. K_1 is a positive constant depending on R, N, b and t_1. $\tilde{A}_s$ and $\tilde{B}_s$ are some polynomials that can be obtained by a recursive relationship.

The asymptotic expansion for $U(a, b, c)$ when $|a| \to \infty$ in the complex plane is considerably more involved, so we prove it step by step, with several lemmas. According to (Volkmer et al., 2016) page 19, when $\Re(b) \geq 1$, $e^{-i\theta} W_2(t, \mu, x) = \beta_1(u) e^{-\frac{1}{2}z^2} z^b F(a, b, z^2) + \beta_2(u) e^{-\frac{1}{2}z^2} z^b U(a, b, z^2)$, where $a = \frac{1}{t}u^2 + \frac{1}{2}b$, $u = te^{i\theta}$, $b = \mu + 1$ and $z = e^{-i\theta}x$. From Olver (Olver, 1956), we have the following expansion:

$$W_2(t, \mu, x) = xK_\mu(tx)\left[\sum_{s=0}^{N-1} \frac{\tilde{A}_s(x)}{t^{2s}} + g_3(t, x)\right] - \frac{x}{t} K_{\mu+1}(tx)\left[\sum_{s=0}^{N-1} \frac{\tilde{B}_s(x)}{t^{2s}} + xh_3(t, x)\right] \tag{3.2.3}$$

where K_v is the modified Bessel function of the second kind, $|g_3| + |h_3| \leq \frac{K_2}{t^{2N}}$ for $0 < |x| \leq R$, $t \geq t_1$. $|\arg z| \leq \frac{3}{2}\pi - \delta$. K_2 is a positive constant depending on R, N, μ and t_1. So we need to discuss the asymptotic behavior of β_1 and β_2. Suppose for now that $\Re(b) \geq 1$; the case that $\Re(b) < 1$ will be dealt with later by transformation. For $\beta(u)$ and U, we have the following two lemmas in (Volkmer et al., 2016).

Lemma 3.2.2. *Suppose $\Re(b) \geq 1$. For every $N = 1, 2, 3, \ldots$, we have*

$$\beta_2(u) = \Gamma(a) 2^{b-2} u^{1-b}\left[1 + 2(1-b)\sum_{s=0}^{N-1} \frac{B_s'(0)}{u^{2s+2}}\right] + \mathcal{O}(u^{-(2N+2)}) \tag{3.2.4}$$

Lemma 3.2.3. *Let $b \in \mathbb{C}$, $\Re(c) > 0$, and $\epsilon > 0$. There is a constant Q independent of a such that $|\Gamma(a)U(a, b, c)| \leq Q$ if $\Re(a) \geq \epsilon$.*

With the help Lemma 3.2.2 and 3.2.3, we are able to show:

Lemma 3.2.4. *Suppose $\Re(b) \geq 1$. $u = te^{i\theta}$, $|\theta| < \frac{\pi}{2} - \delta$, $z > 0$ real, $|z| \leq R$, then for every $q < R$, we have $\beta_1(u) = \mathcal{O}(e^{-tq\cos(\frac{\pi}{2}-\delta)})$ as $|u| = t \to \infty$.*

For $\beta_2(u) e^{-\frac{1}{2}z^2} z^b U(a, b, z^2)$ and $\beta_2(u)$, we have the following two lemmas:

Lemma 3.2.5. *Suppose $\Re(b) \geq 1$, then*

$$\begin{aligned}\beta_2(u)e^{-\frac{1}{2}z^2}z^bU(a,b,z^2) = zK_{b-1}(zu)\left[\sum_{s=0}^{N-1}\frac{\tilde{A}_s(e^{i\theta}z)}{t^{2s}} + g_4(t,\theta,z)\right] \\ - \frac{z}{t}K_b(zu)\left[\sum_{s=0}^{N-1}\frac{\tilde{B}_s(e^{i\theta}z)}{t^{2s}} + zh_4(t,\theta,z)\right]\end{aligned} \tag{3.2.5}$$

where $|g_4| + |h_4| \leq \frac{K_3}{t^{2N}}$ for $0 < |z| \leq R\cos(\frac{\pi}{2} - \delta)/3$, $t \geq t_1$, z real and $|\theta| \leq \frac{\pi}{2} - \delta$. K_3 is a positive constant depending on R, N, b, δ and t_1.

Lemma 3.2.6. *With the same condition in Lemma 3.2.5, for all $N = 1, 2, 3...$, we have:*

$$\frac{\beta_2(u)2^bu^{1-b}}{\Gamma(1+a-b)} = 1 + \mathcal{O}(t^{-2N}) \tag{3.2.6}$$

With Lemma 3.3.2 to Lemma 3.3.6, we have the asymptotic expansion for $U(a,b,z^2)$:

Lemma 3.2.7. *For $\Re(b) > 0$, we have the asymptotic expansion for $U(\frac{1}{4}u^2 + \frac{1}{2}b, b, z^2)$:*

$$\begin{aligned}&\Gamma(1+\frac{1}{4}u^2-\frac{1}{2}b)2^{-b}u^{b-1}e^{-\frac{1}{2}z^2}z^bU(\frac{1}{4}u^2+\frac{1}{2}b,b,z^2) \\ &= zK_{b-1}(zu)\left[\sum_{s=0}^{N-1}\frac{\tilde{A}_s(e^{i\theta}z)}{t^{2s}} + g_5(t,\theta,z)\right] - \frac{z}{t}K_b(zu)\left[\sum_{s=0}^{N-1}\frac{\tilde{B}_s(e^{i\theta}z)}{t^{2s}} + zh_5(t,\theta,z)\right]\end{aligned} \tag{3.2.7}$$

where $|g_5| + |h_5| \leq \frac{K_4}{t^{2N}}$ for $0 < |z| \leq R\cos(\frac{\pi}{2} - \delta)/3$, $u = te^{i\theta}$, $t \geq t_1$, z real and $|\theta| \leq \frac{\pi}{2} - \delta$. K_4 is a positive constant depending on R, N, b, δ, and t_1.

Now we are ready to state the theorem related to the asymptotic expansion of $F(a,b,z^2)$ and $U(a,b,z^2)$ as $|a| \to \infty$ in the complex plane:

Theorem 3.2.8. *Suppose that $\Re(b) > 0$, $u = te^{i\theta}$ with $t > 0$, $N \geq 1$, then*

$$\begin{aligned}F(\frac{1}{4}u^2+\frac{1}{2}b,b,z^2) = \frac{\Gamma(b)}{2^{1-b}u^{b-1}}e^{\frac{1}{2}z^2}z^{1-b}\Big[I_{b-1}(uz)\left(\sum_{s=0}^{N-1}\frac{\tilde{A}_s(e^{i\theta}z)}{t^{2s}} + g_{21}(t,e^{i\theta}z)\right) \\ + \frac{1}{t}I_b(uz)\Big(\sum_{s=0}^{N-1}\frac{\tilde{B}_s(e^{i\theta}z)}{t^{2s}} + e^{i\theta}zh_{21}(t,e^{i\theta}z)\Big)\Big]\end{aligned} \tag{3.2.8}$$

where $|g_{2i}| + |h_{2i}| \leq \frac{K_1}{t^{2N}}$ for $0 < |z| \leq R$, $t \geq t_1$. K_1 is a positive constant only depending on R, N, b and t_1. $\tilde{A}_s$ and $\tilde{B}_s$ are some polynomials that can be obtained by a recursive relationship. Additionally if we have z real, $|\theta| \leq \frac{\pi}{2} - \delta$, then:

$$\begin{aligned}&U(\frac{1}{4}u^2+\frac{1}{2}b,b,z^2) = \frac{2^b}{\Gamma(1+\frac{1}{4}u^2-\frac{1}{2}b)}u^{1-b}e^{\frac{1}{2}z^2}z^{-b}\times \\ &\Big[zK_{b-1}(zu)(\sum_{s=0}^{N-1}\frac{\tilde{A}_s(e^{i\theta}z)}{t^{2s}} + g_5(t,\theta,z)) - \frac{z}{t}K_b(zu)(\sum_{s=0}^{N-1}\frac{\tilde{B}_s(e^{i\theta}z)}{t^{2s}} + zh_5(t,\theta,z))\Big]\end{aligned} \tag{3.2.9}$$

where $|g_5| + |h_5| \leq \frac{K_4}{t^{2N}}$ for $0 < |z| \leq R$. K_4 is a positive constant depending on R, N, b, δ and t_1.

3.3 IDENTIFIABILITY OF UNKNOWN PARAMETERS

Before we discuss how to make statistical inference for the unknown parameters, we should clarify whether those parameters are identifiable. In other words, we must guarantee that different parameters in the parameter space should return us distinguishable distributions of the first passage time T_{y_c}. The model has 5 unknown parameters $(\alpha, \beta, k, y, y_c)$. However, our analysis below shows that not all of them are estimable given only the first passage times. We have the following lemma which states the four identifiable functions of the five parameters.

Lemma 3.3.1. *Given only first passage time observations, the identifiable parameters are*

$$(\alpha, \frac{2\beta}{k^2}, \frac{2\alpha y}{k^2}, \frac{2\alpha y_c}{k^2}).$$

3.4 VALIDITY OF THE INVERSION FORMULA

By Lemma 3.3.1, our parameter space is

$$\mathcal{A} = \{(\alpha_1, \alpha_2, \alpha_3, \alpha_4) : \alpha_i > 0 \text{ for all } i\}$$

with the following connection to the original parameters: $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (\frac{1}{\alpha}, \frac{2\beta}{k^2}, \frac{2\alpha y}{k^2}, \frac{2\alpha y_c}{k^2})$. We first extend s to the complex plane. Let $s = \frac{u^2}{4\alpha_1} + \frac{\alpha_2}{2\alpha_1}$, where $u = te^{i\theta}$. Notice that if we restrict $\Re s \geq \frac{\alpha_2}{2\alpha_1}$, we will have $-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$. Then the expansion of F and U in Theorem 3.2.8 and expansion of I_v, K_v in Equation (A.5.12) and (A.5.13) are all valid. Thus, we can obtain for $\Re\alpha_2 > 0$:

$$\begin{aligned}
&\frac{F(\alpha_1 s, \alpha_2, \alpha_3)}{F(\alpha_1 s, \alpha_2, \alpha_4)} = e^{\frac{1}{2}(\alpha_3 - \alpha_4)}(\frac{\alpha_3}{\alpha_4})^{\frac{1-\alpha_2}{2}} \times \\
&\frac{I_{\alpha_2 - 1}(u\sqrt{\alpha_3})(1 + \mathcal{O}(t^{-2})) + \frac{1}{t} I_{\alpha_2}(u\sqrt{\alpha_3})(e^{-i\theta}\alpha_3^{\frac{3}{2}} + \mathcal{O}(t^{-2}))}{I_{\alpha_2 - 1}(u\sqrt{\alpha_4})(1 + \mathcal{O}(t^{-2})) + \frac{1}{t} I_{\alpha_2}(u\sqrt{\alpha_4})(e^{-i\theta}\alpha_4^{\frac{3}{2}} + \mathcal{O}(t^{-2}))} \\
&= e^{\frac{1}{2}(\alpha_3 - \alpha_4)}(\frac{\alpha_3}{\alpha_4})^{\frac{1-2\alpha_2}{4}} e^{u(\sqrt{\alpha_3} - \sqrt{\alpha_4})}[1 + \mathcal{O}(t^{-1})] \\
&= e^{\frac{1}{2}(\alpha_3 - \alpha_4)}(\frac{\alpha_3}{\alpha_4})^{\frac{1-2\alpha_2}{4}} e^{(t\cos\theta + it\sin\theta)(\sqrt{\alpha_3} - \sqrt{\alpha_4})}[1 + \mathcal{O}(t^{-1})]
\end{aligned} \tag{3.4.1}$$

Meanwhile, for U function we have:

$$\frac{U(\alpha_1 s,\alpha_2,\alpha_3)}{U(\alpha_1 s,\alpha_2,\alpha_4)} = e^{\frac{1}{2}(\alpha_3-\alpha_4)}(\frac{\alpha_3}{\alpha_4})^{\frac{1-\alpha_2}{2}} \times$$
$$\frac{K_{\alpha_2-1}(u\sqrt{\alpha_3})(1+\mathcal{O}(t^{-2})) + \frac{1}{t}K_{\alpha_2}(u\sqrt{\alpha_3})(e^{-i\theta}\alpha_3^{\frac{3}{2}}+\mathcal{O}(t^{-2}))}{K_{\alpha_2-1}(u\sqrt{\alpha_4})(1+\mathcal{O}(t^{-2})) + \frac{1}{t}K_{\alpha_2}(u\sqrt{\alpha_4})(e^{-i\theta}\alpha_4^{\frac{3}{2}}+\mathcal{O}(t^{-2}))} \tag{3.4.2}$$
$$= e^{\frac{1}{2}(\alpha_3-\alpha_4)}(\frac{\alpha_3}{\alpha_4})^{\frac{1-2\alpha_2}{4}} e^{u(\sqrt{\alpha_4}-\sqrt{\alpha_3})}[1+\mathcal{O}(t^{-1})]$$
$$= e^{\frac{1}{2}(\alpha_3-\alpha_4)}(\frac{\alpha_3}{\alpha_4})^{\frac{1-2\alpha_2}{4}} e^{(t\cos\theta+it\sin\theta)(\sqrt{\alpha_4}-\sqrt{\alpha_3})}[1+\mathcal{O}(t^{-1})]$$

Then if $|s| \to \infty$ with $\Re(s) \geq \frac{\alpha_2}{2\alpha_1}$, we know $t \to +\infty$, then $|\frac{F(\alpha_1 s,\alpha_2,\alpha_3)}{F(\alpha_1 s,\alpha_2,\alpha_4)}|$ (with $\alpha_3 < \alpha_4$) and $|\frac{U(\alpha_1 s,\alpha_2,\alpha_3)}{U(\alpha_1 s,\alpha_2,\alpha_4)}|$ (with $\alpha_3 > \alpha_4$) are both exponentially decaying in t. By Theorem 5 on page 195 of (Churchill, 1972), the Bromwich integral inversion formula is valid. Let $\hat{g}(s|\alpha) = \frac{F(\alpha_1 s,\alpha_2,\alpha_3)}{F(\alpha_1 s,\alpha_2,\alpha_4)}$, $\hat{h}(s|\alpha) = \frac{U(\alpha_1 s,\alpha_2,\alpha_3)}{U(\alpha_1 s,\alpha_2,\alpha_4)}$, we can apply the Bromwich integral which yields the continuous pdfs

$$g(t|\alpha) = \frac{1}{2\pi i}\int_{\frac{\alpha_2}{2\alpha_1}-i\infty}^{\frac{\alpha_2}{2\alpha_1}+i\infty} e^{ts}\hat{g}(s|\alpha)ds \quad \text{and} \quad h(t|\alpha) = \frac{1}{2\pi i}\int_{\frac{\alpha_2}{2\alpha_1}-i\infty}^{\frac{\alpha_2}{2\alpha_1}+i\infty} e^{ts}\hat{h}(s|\alpha)ds. \tag{3.4.3}$$

Next, we show the partial derivatives of $\hat{g}(s|\alpha)$ and $\hat{h}(s|\alpha)$ with respect to the parameters also exist and are appropriately bounded; here we have only finished the proof for $\hat{g}$, and will deal with $\hat{h}$ in the future. In order to check the validity of the Bromwich integral for the derivatives, we have to establish a neighbourhood argument, then we need the following lemma:

Lemma 3.4.1. *According to the expansion in Theorem 3.2.8, define* $x(s,\alpha_1,\alpha_2,\alpha_3)$ *by*

$$F(\alpha_1 s,\alpha_2,\alpha_3) = \frac{\Gamma(\alpha_2)}{\pi^{\frac{1}{2}}2^{\frac{3}{2}-\alpha_2}u^{\alpha_2-\frac{1}{2}}} e^{\frac{1}{2}\alpha_3}\alpha_3^{\frac{1-2\alpha_2}{4}} e^{u\sqrt{\alpha_3}}[1+x(s,\alpha_1,\alpha_2,\alpha_3)],$$

where $\frac{1}{4}u^2+\frac{1}{2}\alpha_2 = \alpha_1 s$, $u = te^{i\theta}$, $\theta \in [-\frac{\pi}{4},\frac{\pi}{4}]$, *then in a small neighborhood of* $\alpha_{01} > 0$, $\alpha_{02} > 0$ $\alpha_{03} > 0$ *in the complex plane, there exists* $k_1 > 0$, $k_2 < \infty$ *such that*

$$k_1 \leq |1+x(s,\alpha_1,\alpha_2,\alpha_3)| \leq k_2$$

Define $F_1 = \frac{\partial F(x,y,z)}{\partial x}$, $F_2 = \frac{\partial F(x,y,z)}{\partial y}$, $F_3 = \frac{\partial F(x,y,z)}{\partial z}$, $F_{12} = \frac{\partial^2 F(x,y,z)}{\partial x\partial y}$ ..., $U_1 = \frac{\partial U(x,y,z)}{\partial x}$, ..., $\hat{g}_i = \frac{\partial \hat{g}}{\partial \alpha_i}$, $\hat{g}_{ij} = \frac{\partial^2 \hat{g}}{\partial\alpha_i\partial\alpha_j}$, ..., $\hat{h}_i = \frac{\partial \hat{h}}{\partial \alpha_i}$, $\hat{h}_{ij} = \frac{\partial^2 \hat{h}}{\partial\alpha_i\partial\alpha_j}$, ... , similarly for g, h. Then for those partial derivatives of $\hat{g}$ and $\hat{h}$, we have the following lemma:

Lemma 3.4.2. *In the region $\Re_s \geq \frac{\alpha_2}{2\alpha_1}$, all partial derivatives of $\hat{g}$ and $\hat{h}$ with respect to α decay exponentially as $|s| \to \infty$.*

Now we extend the results to a small closed neighborhood of a value of $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$.

Lemma 3.4.3. *For $\alpha_1 > 0$, $\alpha_2 > 0$, $0 < \alpha_3 < \alpha_4$, in a small closed neighborhood of $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$, with $\Re(s) > t_1$, there exist positive constants A, B, A_i, B_i, A_{ij}, B_{ij}, A_{ijk}, B_{ijk} such that:*

$$|\hat{g}| \leq Ae^{-B\sqrt{|s|}}, \qquad |\hat{g}_i| \leq A_i e^{-B_i\sqrt{|s|}}, \qquad |\hat{g}_{ij}| \leq A_{ij} e^{-B_{ij}\sqrt{|s|}}, \qquad |\hat{g}_{ijk}| \leq A_{ijk} e^{-B_{ijk}\sqrt{|s|}}.$$

So the Bromwich integral can be also applied to the derivative with respect to the parameters:

$$\begin{aligned}
\frac{\partial g(t|\alpha)}{\partial \alpha_i} &= \frac{1}{2\pi i}\int_{\frac{\alpha_2}{2\alpha_1}-i\infty}^{\frac{\alpha_2}{2\alpha_1}+i\infty} e^{ts}\hat{g}_i(s|\alpha)ds \\
\frac{\partial g^2(t|\alpha)}{\partial \alpha_i \partial \alpha_j} &= \frac{1}{2\pi i}\int_{\frac{\alpha_2}{2\alpha_1}-i\infty}^{\frac{\alpha_2}{2\alpha_1}+i\infty} e^{ts}\hat{g}_{ij}(s|\alpha)ds \\
\frac{\partial g^3(t|\alpha)}{\partial \alpha_i \partial \alpha_j \partial \alpha_k} &= \frac{1}{2\pi i}\int_{\frac{\alpha_2}{2\alpha_1}-i\infty}^{\frac{\alpha_2}{2\alpha_1}+i\infty} e^{ts}\hat{g}_{ijk}(s|\alpha)ds
\end{aligned} \tag{3.4.4}$$

3.5 TAIL BEHAVIOR OF DENSITY FUNCTION AND ITS PARTIAL DERIVATIVES WHEN T IS LARGE

We have investigated the tail behavior of the Laplace transform and its derivatives with respect to those identifiable parameters in a small closed neighbourhood, which enables us to use the Bromwich integral to obtain not only the density function, but also its partial derivatives with respect to the parameters. In the next few lemmas, we establish the property for the density function and its partial derivatives when $t \to +\infty$.

From (Slater, 1960), we have $F(a, b, x) = \Gamma(b)e^{x/2}(kx)^{\frac{1}{2}-\frac{1}{2}b}\sum_{n=0}^{\infty} U_n$, where $k = \frac{1}{2}b - a$, $U_n = u_{3n} + u_{3n+1} + u_{3n+2}$, $u_n = B_n(k, \frac{b}{2})(\frac{x}{4k})^{\frac{1}{2}n} J_{b+n-1}(2k^{\frac{1}{2}}x^{\frac{1}{2}})$. J is Bessel function of the first kind, and $B_n(k, \frac{b}{2})$ satisfies the following recurrence relation:

$$(n+1)B_{n+1}(k, \frac{1}{2}b) = (n+b-1)B_{n-1}(k, \frac{1}{2}b) - 2kB_{n-2}(k, \frac{1}{2}b)$$

with $B_0 = 1$, $B_1 = 0$, $B_2 = \frac{1}{2}b$.

Now here is the first lemma for the bound of B_n:

Lemma 3.5.1. *In a closure of* (b^0, x^0)*, there exist* r_1 *such that for* $n \geq 0$*, and* k *large:*

$$|B_{3n}| \leq r_1 k^n, \qquad |B_{3n+1}| \ \leq r_1 k^n, \qquad |B_{3n+2}| \leq 2r_1 k^n$$

According to page 68 of (Slater, 1960), $|J_{b+n-1}(2k^{\frac{1}{2}}x^{\frac{1}{2}})| \leq 1$ when b is real and positive and $n = 1, 2, \cdots$. We can see that in a closure of (b, x), for large k there exists r_2 such that

For $n \geq 1$:

$$|U_n| \leq r_2 |k|^{-\frac{1}{2}n} |\frac{x}{4}|^{\frac{3n}{2}}$$

For $n = 0$:

$$|U_n| \leq |J_{b-1}(2k^{\frac{1}{2}}x^{\frac{1}{2}})| + |\frac{1}{2}b\frac{x}{4k}|$$

Combine the two observations above to get

$$F(a,b,x) = \Gamma(b)\exp(\frac{1}{2}x)(kx)^{\frac{1}{2}-\frac{1}{2}b}(J_{b-1}(2k^{\frac{1}{2}}x^{\frac{1}{2}}) + R^1(a,b,x)) \tag{3.5.1}$$

with

$$|R^1(a,b,x)| \leq k_1 k^{-\frac{1}{2}}$$

On the other hand, $J_v(\xi) = \sqrt{\frac{2}{\pi\xi}}\cos(\xi - \frac{1}{2}\pi v - \frac{1}{4}\pi)(1 + \mathcal{O}(|\xi^{-1}|))$, where $\mathcal{O}(|\xi^{-1}|)$ can be uniform in a neighbourhood of (v, ξ). So we have the lemma below:

Lemma 3.5.2. *In a neighbourhood of* (b^0, x^0)*, when* $-a$ *is large,* $F(a, b, x)$ *can be written as:*

$$F(a,b,x) = \Gamma(b)e^{x/2}(kx)^{\frac{1}{2}-\frac{1}{2}b}\left[(\pi x^{\frac{1}{2}}k^{\frac{1}{2}})^{-\frac{1}{2}}\cos(w) + R^2(a,b,x)\right] \tag{3.5.2}$$

where

$$w = 2k^{\frac{1}{2}}x^{\frac{1}{2}} - \frac{1}{2}b\pi + \frac{1}{4}\pi, \qquad k = \frac{b}{2} - a, \quad |R^2(a,b,x)| \leq k_2 k^{-\frac{1}{2}}.$$

Next we need to deal with the derivatives, and have the following lemma:

Lemma 3.5.3. *In a closure of* (b^0, x^0)*, when* $-a$ *is large,* $\exists\ k_3$ *such that:*

$$\begin{aligned}\frac{\partial F(a,b,x)}{\partial a} =& \Gamma(b)e^{x/2}(kx)^{\frac{1}{2}-\frac{1}{2}b-1}(-x)(\frac{1}{2} - \frac{b}{2})\Big((\pi x^{\frac{1}{2}}k^{\frac{1}{2}})^{-\frac{1}{2}}\cos(w) + R^2(a,b,x)\Big) \\ &+ \Gamma(b)e^{x/2}(kx)^{\frac{1}{2}-\frac{1}{2}b}\Big(x^{\frac{1}{2}}k^{-\frac{1}{2}}((\pi x^{\frac{1}{2}}k^{\frac{1}{2}})^{-\frac{1}{2}}\sin(w) + R^3(a,b,x)\Big)\end{aligned} \tag{3.5.3}$$

where R^2 *is the same in Lemma 3.5.2 and* $|R^3(a,b,x)| \leq k_3 k^{-1}$.

Before we establish the tail behavior, we still need a lemma to deal with the asymptotic of a-zeros of $F(a,b,x)$ in a neighbourhood of (b^0, x^0):

Lemma 3.5.4. *Let a_p be the pth zero of $F(a,b,x)$, then $\exists$ p_0, c_1, c_2, $l_1 < p_0$, $l_2 < p_0$ such that in a neighbourhood of (b^0, x^0), when $p > p_0$,*

$$-a_0 < c_1(p\pi - l_1)^2 \le -a_p \le c_2(p\pi - l_2)^2 \qquad (3.5.4)$$

Now we can establish the tail behavior of density function $g(t|\alpha)$. Since α_1 is a scale parameter, we can let $\alpha_1 = 1$. The inversion of Laplace transform can be written as:

$$g(t|\alpha) = \sum_{p=0}^{\infty} A_p(\alpha)\exp(s_p t) \qquad (3.5.5)$$

where s_p are decreasing zeros of $F(s, \alpha_2, \alpha_4)$, and $A_p(\alpha) = \frac{F(s_p,\alpha_2,\alpha_3)}{\frac{\partial}{\partial s}F(s_p,\alpha_2,\alpha_4)}$. We can prove the following lemma:

Lemma 3.5.5. *In a small neighbourhood of $(\alpha_2^0, \alpha_3^0, \alpha_4^0)$, there exist t_1, k_4 and k_5 positive such that when $t \ge t_1$:*

$$k_5 A_0(\alpha)\exp(s_0 t) \le g(t|\alpha) \le k_4 A_0(\alpha)\exp(s_0 t). \qquad (3.5.6)$$

Next, we come to deal with the derivatives. Consider α_3 first:

$$\frac{\frac{\partial}{\partial \alpha_3}F(s_p,\alpha_2,\alpha_3)}{\frac{\partial}{\partial s}F(s_p,\alpha_2,\alpha_4)} = \frac{\frac{s_p}{\alpha_2}F(s_p+1,\alpha_2+1,\alpha_3)}{\frac{\partial}{\partial s}F(s_p,\alpha_2,\alpha_4)}$$

With Lemma (3.5.2) again and a similar argument in Lemma (3.5.5), it is not hard to see that $\exists$ r_{11} and p_{10} such that when $p \ge p_0$ in a neighbourhood of $(\alpha_2^0, \alpha_3^0, \alpha_4^0)$:

$$\left|\frac{\frac{\partial}{\partial \alpha_3}F(s_p,\alpha_2,\alpha_3)}{F(s_p,\alpha_2,\alpha_4)}\right| \le r_{11}(-s_p) \qquad (3.5.7)$$

So we have uniform convergence for $\sum_{p=0}^{\infty} \frac{\partial A_p(\alpha)}{\partial \alpha_3}\exp(s_p t)$ in a neighbourhood of α_3^0 when $t \ge t_1$, thus

$$\frac{\partial}{\partial \alpha_3}g(t|\alpha) = \sum_{p=0}^{\infty}\frac{A_p(\alpha)}{\partial \alpha_3}\exp(s_p t) \qquad (3.5.8)$$

and

$$\left|\frac{\partial}{\partial \alpha_3}g(t|\alpha)\right|| \le r_{11}\left|\frac{\partial A_0(\alpha)}{\partial \alpha_3}\right|\exp(s_0 t) \qquad (3.5.9)$$

To deal with the derivatives with respect to α_2 and α_4 we need further investigation, since s_p is a function of (α_2, α_4). Returning to the form

$$F(a,b,x) = \Gamma(b)e^{x/2}(kx)^{\frac{1}{2}-\frac{1}{2}b}\sum_{n=0}^{\infty}U_n$$

we have

$$\begin{aligned}\frac{\partial^2 F}{\partial a^2} =&\Gamma(b)e^{x/2}\left(\frac{1-b}{2}\right)\left(\frac{1-b}{2}-1\right)(kx)^{\frac{1-b}{2}-2}(x^2)\sum_{n=0}^{\infty}U_n\\ &+2\Gamma(b)e^{x/2}\left(\frac{1-b}{2}\right)(kx)^{\frac{1-b}{2}-1}(-x)\frac{\partial\sum_{n=0}^{\infty}U_n}{\partial a}\\ &+\Gamma(b)e^{x/2}(kx)^{\frac{1-b}{2}}\frac{\partial^2\sum_{n=0}^{\infty}U_n}{\partial a^2}\end{aligned} \tag{3.5.10}$$

Continuing with similar calculations in Equation (D.3.1), we have for $n \geq 1$:

$$\begin{aligned}\frac{\partial^2 u_n}{\partial a^2} =&\frac{\partial^2 B_n(k,\frac{b}{2})}{\partial a^2}(\frac{x}{4k})^{\frac{1}{2}n}J_{b+n-1}(2k^{\frac{1}{2}}x^{\frac{1}{2}})\\ &+\frac{\partial B_n(k,\frac{b}{2})}{\partial a}(\frac{x}{4})^{\frac{1}{2}n}k^{-\frac{n}{2}-1}(\frac{n}{2})J_{b+n-1}(2k^{\frac{1}{2}}x^{\frac{1}{2}})\\ &+\frac{\partial B_n(k,\frac{b}{2})}{\partial a}(\frac{x}{4k})^{\frac{1}{2}n}(-x^{\frac{1}{2}}k^{-\frac{1}{2}})\frac{\partial J_{b+n-1}(2k^{\frac{1}{2}}x^{\frac{1}{2}})}{\partial 2k^{\frac{1}{2}}x^{\frac{1}{2}}}(2k^{\frac{1}{2}}x^{\frac{1}{2}})\\ &+\frac{\partial B_n(k,\frac{b}{2})}{\partial a}(\frac{x}{4})^{\frac{1}{2}n}k^{-\frac{n}{2}-1}(\frac{n}{2})J_{b+n-1}(2k^{\frac{1}{2}}x^{\frac{1}{2}})\\ &+B_n(k,\frac{b}{2})(\frac{x}{4})^{\frac{1}{2}n}(-\frac{n}{2}-1)(\frac{n}{2})k^{-\frac{n}{2}-2}J_{b+n-1}(2k^{\frac{1}{2}}x^{\frac{1}{2}})\\ &+B_n(k,\frac{b}{2})(\frac{x}{4})^{\frac{1}{2}n}(\frac{n}{2})k^{-\frac{n}{2}-1}(-x^{\frac{1}{2}}k^{-\frac{1}{2}})\frac{\partial J_{b+n-1}(2k^{\frac{1}{2}}x^{\frac{1}{2}})}{\partial 2k^{\frac{1}{2}}x^{\frac{1}{2}}}(2k^{\frac{1}{2}}x^{\frac{1}{2}})\\ &+\frac{\partial B_n(k,\frac{b}{2})}{\partial a}(\frac{x}{4k})^{\frac{1}{2}n}(-x^{\frac{1}{2}}k^{-\frac{1}{2}})\frac{\partial J_{b+n-1}(2k^{\frac{1}{2}}x^{\frac{1}{2}})}{\partial 2k^{\frac{1}{2}}x^{\frac{1}{2}}}(2k^{\frac{1}{2}}x^{\frac{1}{2}})\\ &+B_n(k,\frac{b}{2})(\frac{x}{4})^{\frac{1}{2}n}k^{-\frac{n}{2}-1}(\frac{n}{2})(-x^{\frac{1}{2}}k^{-\frac{1}{2}})\frac{\partial J_{b+n-1}(2k^{\frac{1}{2}}x^{\frac{1}{2}})}{\partial 2k^{\frac{1}{2}}x^{\frac{1}{2}}}(2k^{\frac{1}{2}}x^{\frac{1}{2}})\\ &+B_n(k,\frac{b}{2})(\frac{x}{4k})^{\frac{1}{2}n}(x^{\frac{1}{2}}k^{-\frac{3}{2}})(\frac{1}{2})\frac{\partial J_{b+n-1}(2k^{\frac{1}{2}}x^{\frac{1}{2}})}{\partial 2k^{\frac{1}{2}}x^{\frac{1}{2}}}(2k^{\frac{1}{2}}x^{\frac{1}{2}})\\ &+B_n(k,\frac{b}{2})(\frac{x}{4k})^{\frac{1}{2}n}(xk^{-1})\frac{\partial^2 J_{b+n-1}(2k^{\frac{1}{2}}x^{\frac{1}{2}})}{\partial(2k^{\frac{1}{2}}x^{\frac{1}{2}})^2}(2k^{\frac{1}{2}}x^{\frac{1}{2}})\end{aligned} \tag{3.5.11}$$

Next we need to investigate the order of each term in Equation (3.5.11) for $3n$, $3n+1$ and $3n+2$; here we consider $3n$, which dominate $3n+1$ and $3n+2$: The first term is can be bounded by some term of order $(n^3k^{n-2})(\frac{x}{4k})^{\frac{3}{2}n}$; the second and forth terms can be bounded by some term of order $n^3k^{n-2}(\frac{x}{4k})^{\frac{3}{2}n}$; the third and seventh terms can be bounded by some term of order

$n^2k^{n-\frac{3}{2}}(\frac{x}{4k})^{\frac{3}{2}n}(1+\frac{n}{k^{\frac{1}{2}}})$; the fifth term can be bounded by some term of order $n^2k^{n-2}(\frac{x}{4k})^{\frac{3}{2}n}$; the sixth and eighth terms are of order $nk^{n-\frac{3}{2}}(\frac{x}{4k})^{\frac{3}{2}n}(1+\frac{n}{k^{\frac{1}{2}}})$; the ninth term is of order $k^{n-\frac{3}{2}}(\frac{x}{4k})^{\frac{3}{2}n}(1+\frac{n}{k^{\frac{1}{2}}})$; the tenth term is of order $k^{n-1}(\frac{x}{4k})^{\frac{3}{2}n}(1+\frac{2n}{k^{\frac{1}{2}}}+\frac{n^2}{k})$. With a similar analysis in Equation (A.12.2), for $n \geq 1$ and k large, $\exists\ r_{12}$, r_{13} and r_{14} such that:

$$\left|\frac{\partial^2 U_n}{\partial a^2}\right| \leq r_{12}(n^3)(\frac{x}{4k})^{\frac{3}{2}n}k^{n-2} + r_{13}n^2(\frac{x}{4k})^{\frac{3}{2}n}k^{n-\frac{3}{2}} + r_{14}k^{n-1}(\frac{x}{4k})^{\frac{3}{2}n} \tag{3.5.12}$$

By the observations above, we can prove the following Lemma for $\frac{\partial^2 F}{\partial a^2}$:

Lemma 3.5.6. *In a closure of (b^0, x^0), when k is large, we have the following expansion for $\frac{\partial^2 F}{\partial a^2}$:*

$$\begin{aligned}\frac{\partial^2 F}{\partial a^2} &= \Gamma(b)e^{x/2}\left(\frac{1-b}{2}\right)\left(\frac{1-b}{2}-1\right)(kx)^{\frac{1-b}{2}-2}(x^2)\Big((\pi x^{\frac{1}{2}}k^{\frac{1}{2}})^{-\frac{1}{2}}\cos(w) + R^2(a,b,x)\Big)\\ &+ 2\Gamma(b)e^{x/2}\left(\frac{1-b}{2}\right)(kx)^{\frac{1-b}{2}-1}(-x)\Big(x^{\frac{1}{2}}k^{-\frac{1}{2}}(\pi x^{\frac{1}{2}}k^{\frac{1}{2}})^{-\frac{1}{2}}\sin(w) + R^3(a,b,x)\Big)\\ &+ \Gamma(b)e^{x/2}(kx)^{\frac{1-b}{2}}\Big(-\sqrt{\frac{x^{\frac{1}{2}}}{\pi k^{\frac{3}{2}}}}\cos(w) + R^4(a,b,x)\Big)\end{aligned} \tag{3.5.13}$$

where $\exists\ k_4$ such that $|R^4(a,b,x)| \leq k_4k^{-1}$.

Now we are able to handle the derivatives with respect to α_4.

$$\frac{\partial A_p(\alpha)}{\alpha_4} = \frac{\frac{\partial F}{\partial s}(s_p,\alpha_2,\alpha_3)\frac{\partial s_p}{\partial \alpha_4}}{\frac{\partial F}{\partial s}(s_p,\alpha_2,\alpha_4)} - \frac{F(s_p,\alpha_2,\alpha_3)}{\frac{\partial F}{\partial s}(s_p,\alpha_2,\alpha_4)} \times \frac{\frac{\partial s_p}{\partial \alpha_4}\frac{\partial^2 F}{\partial s^2}(s_p,\alpha_2,\alpha_4) + \frac{\partial^2 F}{\partial s\partial \alpha_4}(s_p,\alpha_2,\alpha_4)}{\frac{\partial F}{\partial s}(s_p,\alpha_2,\alpha_4)} \tag{3.5.14}$$

$$\frac{\partial^2 F}{\partial s\partial \alpha_4}(s_p,\alpha_2,\alpha_4) = \frac{1}{\alpha_2}F(s_p+1,\alpha_2+1,\alpha_4) + \frac{s_p}{\alpha_2}\frac{\partial}{\partial s}F(s_p+1,\alpha_2+1,\alpha_4)$$

With Lemma (3.5.2) and (3.5.3), we know that in a closure of $(\alpha_2^0, \alpha_3^0, \alpha_4^0)$, $\exists\ r_{15}$ such that when $-s_p$ is large we have:

$$\left|\frac{\frac{\partial^2 F}{\partial s\partial \alpha_4}(s_p,\alpha_2,\alpha_4)}{\frac{\partial F}{\partial s}(s_p,\alpha_2,\alpha_4)}\right| \leq r_{15}(-s_p)^{\frac{1}{2}} \tag{3.5.15}$$

On the other hand, because $F(s_p,\alpha_2,\alpha_4) = 0$, differentiate it with respect to α_4 we have:

$$\frac{\partial s_p}{\partial \alpha_4}\frac{\partial F}{\partial s}(s_p,\alpha_2,\alpha_4) + \frac{s_p}{\alpha_2}F(s_p+1,\alpha_2+1,\alpha_4) = 0$$

So

$$\frac{\partial s_p}{\partial \alpha_4} = -\frac{\frac{s_p}{\alpha_2}F(s_p+1,\alpha_2+1,\alpha_4)}{\frac{\partial F}{\partial s}(s_p,\alpha_2,\alpha_4)} \tag{3.5.16}$$

When $-s_p$ is large, in a neighbourhood of $(\alpha_2^0, \alpha_3^0, \alpha_4^0)$, $\exists\, r_{16}$ such that:

$$\left|\frac{\partial s_p}{\partial \alpha_4}\right| \leq r_{16}(-s_p) \tag{3.5.17}$$

By Equation (3.5.15), (3.5.16), (3.5.18) and Lemma (3.5.6), when $-s_p$ is large, we have a good bound for $\frac{\partial A_p(\alpha)}{\alpha_4}$ in a neighbourhood of parameters:

$$\left|\frac{\partial A_p(\alpha)}{\alpha_4}\right| \leq r_{17}(-s_p)^{\frac{3}{2}} \tag{3.5.18}$$

With Equation (3.5.17) and a similar argument in Lemma (3.5.5), in a neighbourhood of $(\alpha_2^0, \alpha_3^0, \alpha_4^0)$, we have:

$$\frac{\partial g}{\partial \alpha_4}(t|\alpha) = \sum_{p=0}^{\infty}\left(\frac{\partial A_p(\alpha)}{\partial \alpha_4}\exp(s_p t) + t\frac{\partial s_p}{\partial \alpha_4}\exp(s_p t)\right) \tag{3.5.19}$$

$\exists\, k_6$ such that when $t \geq t_1$ in Lemma (3.5.5):

$$\left|\frac{\partial g}{\partial \alpha_4}(t|\alpha)\right| \leq k_6\left[\left|\frac{\partial A_0(\alpha)}{\partial \alpha_4}\right|\exp(s_0 t) + t\left|\frac{\partial s_0}{\partial \alpha_4}\right|\exp(s_0 t)\right] \tag{3.5.20}$$

As to α_2, since we can not differentiate $F(s_p, \alpha_2, \alpha_4)$ directly with respect to α_2, it is more complicated. We will use the integral representation of $F(a, b, x)$, so we have to modify Lemma (3.5.2) and Lemma (3.5.3).

Lemma 3.5.7. *In a neighbourhood of $b > 0$, when $k = \frac{b}{2} - a$ is large, for $x > 0$ bounded, we have the following expansion for $F(a, b, x)$:*

$$F(a,b,x) = \Gamma(b)e^{x/2}(kx)^{\frac{1-b}{2}}\left((\pi x^{\frac{1}{2}}k^{\frac{1}{2}})^{-\frac{1}{2}}(\cos w + R^5(a,b,x)) + R^1(a,b,x)\right) \tag{3.5.21}$$

where $|R^5(a,b,x)| \leq \frac{k_7}{(2x^{\frac{1}{2}})^{|b-1|+\frac{3}{2}}k^{\frac{1}{2}}}$, $|R^1| \leq \frac{k_8}{k^{\frac{1}{2}}}$.

Lemma 3.5.8. *In a neighbourhood of $b > 0$, when $k = \frac{b}{2} - a$ is large, for $x > 0$ bounded, we have the following expansion for $\frac{\partial F(a,b,x)}{\partial a}$:*

$$\begin{aligned}\frac{\partial F(a,b,x)}{\partial a} =& \Gamma(b)e^{x/2}(kx)^{\frac{1-b}{2}-1}(-x)\left(\frac{1-b}{2}\right)\left(\left((\pi x^{\frac{1}{2}}k^{\frac{1}{2}})^{-\frac{1}{2}}(\cos w + R^5(a,b,x)) + R^1(a,b,x)\right)\right)\\ &+\Gamma(b)e^{x/2}(kx)^{\frac{1-b}{2}}\left(x^{\frac{1}{2}}k^{-\frac{1}{2}}(\pi x^{\frac{1}{2}}k^{\frac{1}{2}})^{-\frac{1}{2}}\Big(\sin(w) + R^6(a,b,x)\Big) + R^7(a,b,x)\right)\end{aligned} \tag{3.5.22}$$

R_1 and R_5 are the same in previous lemmas. We also have

$$|R^6(a,b,x)| \leq \frac{k_{10}}{(2x^{\frac{1}{2}})^{|b-1|+\frac{5}{2}}k^{\frac{1}{2}}}$$

and $|R^7(a,b,x)| \leq \frac{k_{11}}{k}$.

Now we come to the derivative with respect to α_2.

$$\begin{aligned}\frac{\partial A_p(\alpha)}{\alpha_2}=&\frac{\frac{\partial F}{\partial s}(s_p,\alpha_2,\alpha_3)\frac{\partial s_p}{\partial\alpha_2}+\frac{\partial F}{\partial\alpha_2}(s_p,\alpha_2,\alpha_3)}{\frac{\partial F}{\partial s}(s_p,\alpha_2,\alpha_4)}\\&-\frac{F(s_p,\alpha_2,\alpha_3)}{\frac{\partial F}{\partial s}(s_p,\alpha_2,\alpha_4)}\times\frac{\frac{\partial s_p}{\partial\alpha_2}\frac{\partial^2 F}{\partial s^2}(s_p,\alpha_2,\alpha_4)+\frac{\partial^2 F}{\partial s\partial\alpha_2}(s_p,\alpha_2,\alpha_4)}{\frac{\partial F}{\partial s}(s_p,\alpha_2,\alpha_4)}\end{aligned}\tag{3.5.23}$$

With the integral representation of $F(a,b,x)$ for $b>c>0$:

$$F(a,b,x)=\frac{1}{\Gamma(b-c)}\int_0^1 F(a,c,xt)t^{c-1}(1-t)^{b-c-1}dt$$

So

$$\frac{\partial F}{\partial b}(a,b,x)=-\psi(b-c)F(a,b,x)+\frac{1}{\Gamma(b-c)}\int_0^1 F(a,c,xt)t^{c-1}(1-t)^{b-c-1}\ln(1-t)dt$$

From Lemma (3.5.7), we know that in a neighborhood of $(\alpha_2^0,\alpha_3^0,\alpha_4^0)$:

$$\begin{aligned}\frac{\partial F}{\partial\alpha_2}(s_p,\alpha_2,\alpha_3)=&-\psi(\alpha_2-c)F(s_p,\alpha_2,\alpha_4)\\&+\frac{1}{\Gamma(\alpha_2-c)}\int_0^1\Gamma(c)\exp(\frac{\alpha_4 t}{2})(\frac{c}{2}-s_p)^{\frac12-\frac c2}(\alpha_4 t)^{\frac12-\frac c2}\\&\times\left((\frac{1}{\pi(\alpha_4 t)^{\frac12}(\frac c2-s_p)^{\frac12}})^{\frac12}(\cos w_t+R^5(s_p,c,\alpha_4 t))+R^1(s_p,c,\alpha_4 t)\right)\\&\times t^{c-1}(1-t)^{b-c-1}\ln(1-t)dt\\=&-\psi(\alpha_2-c)F(s_p,\alpha_2,\alpha_4)+\frac{1}{\Gamma(\alpha_2-c)}\\&\Big(\int_0^1\Gamma(c)\exp(\frac{\alpha_4 t}{2})(\frac c2-s_p)^{\frac14-\frac c2}\alpha_4^{\frac14-\frac c2}\pi^{-\frac14}\cos w_t t^{\frac c2-\frac34}(1-t)^{b-c-1}\ln(1-t)dt\\&+\int_0^1\Gamma(c)\exp(\frac{\alpha_4 t}{2})(\frac c2-s_p)^{\frac14-\frac c2}\alpha_4^{\frac14-\frac c2}\pi^{-\frac14}R^5(s_p,c,\alpha_4 t)t^{\frac c2-\frac34}(1-t)^{b-c-1}\ln(1-t)dt\\&+\int_0^1\Gamma(c)\exp(\frac{\alpha_4 t}{2})(\frac c2-s_p)^{\frac12-\frac c2}\alpha_4^{\frac12-\frac c2}R^1(s_p,c,\alpha_4 t)t^{\frac c2-\frac12}(1-t)^{b-c-1}\ln(1-t)dt\Big)\end{aligned}\tag{3.5.24}$$

Here we need to be cautious about the order of t to make sure the integral is always finite. We only consider the term involving R^5:

$$\begin{aligned}&\left|\int_0^1\Gamma(c)\exp(\frac{\alpha_4 t}{2})(\frac c2-s_p)^{\frac14-\frac c2}\alpha_4^{\frac14-\frac c2}\pi^{-\frac14}R^5(s_p,c,\alpha_4 t)t^{\frac c2-\frac34}(1-t)^{b-c-1}\ln(1-t)dt\right|\\\le&k_7\int_0^1|\Gamma(c)\exp(\frac{\alpha_4 t}{2})(\frac c2-s_p)^{-\frac14-\frac c2}\alpha_4^{-\frac12-\frac c2-\frac{|c-1|}{2}}\pi^{-\frac14}2^{-|c-1|-\frac32}t^{\frac c2-\frac32-\frac{|c-1|}{2}}(1-t)^{b-c-1}\ln(1-t)|dt\\\le&r_{17}(\frac c2-s_p)^{-\frac14-\frac c2}\end{aligned}\tag{3.5.25}$$

By both Equation (3.5.25) and (3.5.26), we know that:

$$|\frac{\partial F}{\partial \alpha_2}(s_p, \alpha_2, \alpha_3)| \le r_{18}(\frac{\alpha_2}{2} - s_p)^{\frac{1}{4}-\frac{\alpha_2}{2}} + r_{19}(\frac{c}{2} - s_p)^{\frac{1}{4}-\frac{c}{2}} \tag{3.5.26}$$

Since

$$\frac{\partial s_p}{\partial \alpha_2}\frac{\partial F}{\partial s}(s_p, \alpha_2, \alpha_4) + \frac{\partial F}{\partial \alpha_2}(s_p, \alpha_2, \alpha_4) = 0$$

we have

$$\frac{\partial s_p}{\partial \alpha_2} = -\frac{\frac{\partial F}{\partial \alpha_2}(s_p, \alpha_2, \alpha_4)}{\frac{\partial F}{\partial s}(s_p, \alpha_2, \alpha_4)}. \tag{3.5.27}$$

On the other hand, we need a bound for $\frac{\partial^2 F}{\partial s \partial \alpha_2}(s_p, \alpha_2, \alpha_4)$:

$$\begin{aligned}\frac{\partial^2 F}{\partial s \partial \alpha_2}(s_p, \alpha_2, \alpha_4) =& -\psi(\alpha_2 - c)\frac{\partial F}{\partial s}(s_p, \alpha_2, \alpha_4)\\ &+\frac{1}{\Gamma(\alpha_2 - c)}\int_0^1 \frac{\partial F}{\partial s}(s_p, c, \alpha_4 t)t^{c-1}(1-t)^{\alpha_2-c-1}\ln(1-t)dt\end{aligned} \tag{3.5.28}$$

By Lemma (3.5.8) and a similar procedure dealing with $\frac{\partial F}{\partial \alpha_2}(s_p, \alpha_2, \alpha_3)$, we have:

$$\begin{aligned}&\frac{\partial^2 F}{\partial s \partial \alpha_2}(s_p, \alpha_2, \alpha_4)\\ =& -\psi(\alpha_2 - c)\frac{\partial F}{\partial s}(s_p, \alpha_2, \alpha_4) + \frac{1}{\Gamma(\alpha_2 - c)}\int_0^1 t^{c-1}(1-t)^{\alpha_2-c-1}\ln(1-t)\\ &\times\Bigg(\Big(-\Gamma(c)\exp(\frac{\alpha_4 t}{2})\pi^{-\frac{1}{2}}(\frac{1}{2}-\frac{c}{2})\alpha_4^{\frac{1}{4}-\frac{c}{2}}t^{\frac{1}{4}-\frac{c}{2}}(\frac{c}{2}-s_p)^{\frac{1}{4}-\frac{c}{2}-1}(\cos w_t + R^5)\Big)\\ &+\Big(-\Gamma(c)\exp(\frac{\alpha_4 t}{2})(\frac{1}{2}-\frac{c}{2})\alpha_4^{\frac{1}{2}-\frac{c}{2}}t^{\frac{1}{4}-\frac{c}{2}}(\frac{c}{2}-s_p)^{\frac{1}{2}-\frac{c}{2}-1}R^1\Big)\\ &+\Big(\Gamma(c)\exp(\frac{\alpha_4 t}{2})\pi^{-\frac{1}{2}}\alpha_4^{\frac{3}{4}-\frac{c}{2}}t^{\frac{3}{4}-\frac{c}{2}}(\frac{c}{2}-s_p)^{-\frac{1}{4}-\frac{c}{2}}(\sin w_t + R^6)\Big)\\ &+\Big(\Gamma(c)\exp(\frac{\alpha_4 t}{2})\alpha_4^{\frac{1}{2}-\frac{c}{2}}t^{\frac{1}{2}-\frac{c}{2}}(\frac{c}{2}-s_p)^{\frac{1}{2}-\frac{c}{2}}R^7\Big)\Bigg)dt\end{aligned} \tag{3.5.29}$$

Analyze Equation (3.5.29) term by term we have:

$$|\frac{\partial^2 F}{\partial s \partial \alpha_2}(s_p, \alpha_2, \alpha_4)| \le r_{20}(\frac{\alpha_2}{2} - s_p)^{-\frac{1}{4}-\frac{\alpha_2}{2}} + r_{21}(\frac{c}{2} - s_p)^{-\frac{1}{4}-\frac{c}{2}} \tag{3.5.30}$$

Through the observations above, we can obtain that:

$$|\frac{\partial A_p(\alpha)}{\alpha_2}| \le r_{22}(-s_p)^{\frac{3}{2}+2(\sup(\alpha_2)-\inf(\alpha_2))} \tag{3.5.31}$$

So we have

$$\frac{\partial g(t|\alpha)}{\partial \alpha_2} = \sum_{p=0}^{\infty} \Big(\frac{\partial A_p(\alpha)}{\partial \alpha_2} \exp(s_p t) + t \frac{\partial s_p}{\partial \alpha_2} \exp(s_p t) \Big) \tag{3.5.32}$$

and $\exists\ k_{12}$ such that when $t \geq t_1$ in Lemma 5:

$$|\frac{\partial g}{\partial \alpha_2}(t|\alpha)| \leq k_{12} \Big(|\frac{\partial A_0(\alpha)}{\partial \alpha_4}| \exp(s_0 t) + t |\frac{\partial s_0}{\partial \alpha_4}| \exp(s_0 t) \Big) \tag{3.5.33}$$

Then we obtain our concluding lemma for the tail behavior of density function $g(t|\alpha)$ when $t \to \infty$:

Lemma 3.5.9. *In a closure of* $(\alpha_1^0, \alpha_2^0, \alpha_3^0, \alpha_4^0)$, $\exists\ k_4$, t_1, q_{11}, q_{12}, q_{21}, q_{22}, q_3, q_{41} *and* q_{42} *positive such that when* $t \geq t_1$*:*

$$g(t|\alpha) \leq \frac{k_4 A_0(\alpha)}{\alpha_1} \exp(\frac{s_0 t}{\alpha_1}) \tag{3.5.34}$$

$$|\frac{\frac{\partial g(t|\alpha)}{\partial \alpha_1}}{g(t|\alpha)}| \leq q_{11} + q_{12} t \tag{3.5.35}$$

$$|\frac{\frac{\partial g(t|\alpha)}{\partial \alpha_2}}{g(t|\alpha)}| \leq q_{21} + q_{22} t \tag{3.5.36}$$

$$|\frac{\frac{\partial g(t|\alpha)}{\partial \alpha_3}}{g(t|\alpha)}| \leq q_3 \tag{3.5.37}$$

$$|\frac{\frac{\partial g(t|\alpha)}{\partial \alpha_4}}{g(t|\alpha)}| \leq q_{41} + q_{42} t \tag{3.5.38}$$

where k_4, $A_0(\alpha)$, s_0 *are the same with Lemma (3.5.5).*

3.6 TAIL BEHAVIOR OF DENSITY FUNCTION AND ITS PARTIAL DERIVATIVES WHEN T IS SMALL

By far, we have derived the property for $g(t|\alpha)$ and $|\frac{\frac{\partial g(t|\alpha)}{\partial \alpha_i}}{g(t|\alpha)}|$ in a neighbourhood of parameters when $t \to +\infty$. In order to check the regularity conditions of maximum likelihood estimation, we would like to investigate those asymptotic when $t \to 0^+$. An intuitive solution is to use the Tauberian theorem, which links the asymptotic of Laplace transform and probability measure. However, the Tauberian theorem only deals with single measure, but we need a neighborhood argument. Here we give a modification of De Bruijn's Tauberian's theorem:

Theorem 3.6.1. *Let μ_n be a sequence of measures supported on $(0,\infty)$, $M_n(\lambda) = \int_0^\infty e^{-\lambda x} d\mu_n(x)$. $\phi \in R_\alpha(0^+)$ (function of regular variation with index $\alpha < 0$).$\psi(\lambda) = \phi(\lambda)/\lambda \in R_{\alpha-1}(0^+)$.Then for a sequence of $x_n \to 0^+$ and $B > 0$: $\forall \rho > 0$, $\lim_n \rho x_n \log(M_n(\psi(\rho x_n))) = h(B) = -(1-\alpha)(\frac{B}{-\alpha})^{\frac{\alpha}{\alpha-1}}$ if and only if $\forall \rho > 0$, $\lim_n \rho x_n \log \mu_n(0, \frac{1}{\phi(\rho x_n)}) = -B$.*

We would like to use three lemmas to help us prove Theorem (3.6.1):

Lemma 3.6.2. *If $\forall \rho > 0$, $\limsup_n \rho x_n \log \mu_n(0, \frac{1}{\phi(\rho x_n)}) \leq -B$, $\lambda_0 = (\frac{B}{-\alpha})^{\frac{1}{\alpha-1}} > 0$, then $\forall \rho > 0$*

$$\limsup_n \rho x_n \log \int_{\frac{1}{\phi(\frac{\rho x_n}{\xi})}}^{+\infty} e^{-x\psi(\rho x_n)} d\mu_n(x) \leq -B\xi - \xi^\alpha \quad 0 < \xi < \lambda_0. \tag{3.6.1}$$

$$\limsup_n \rho x_n \log \int_{-\infty}^{\frac{1}{\phi(\frac{\rho x_n}{\xi})}} e^{-x\psi(\rho x_n)} d\mu_n(x) \leq -B\xi - \xi^\alpha \quad \lambda_0 < \xi < +\infty. \tag{3.6.2}$$

Lemma 3.6.3. *If $\forall \rho > 0$, $\limsup_n \rho x_n \log \mu_n(0, \frac{1}{\phi(\rho x_n)}) \leq -B$, then $\limsup_n \rho x_n \log M_n(\psi(\rho x_n)) \leq h(B)$.*

Lemma 3.6.4. *If $\forall \rho > 0$, $\limsup_n \rho x_n \log \mu_n(0, \frac{1}{\phi(\rho x_n)}) \leq -B$, with $\liminf_n \rho x_n \log M_n(\psi(\rho x_n)) \geq C > -\infty$. Then $\liminf_n \rho x_n \log \mu_n(0, \frac{1}{\phi(\rho x_n)}) \geq -B\frac{\lambda_2}{\lambda_1}$. Where $\lambda_2 \geq \lambda_1$ are two roots of $-B\lambda - \lambda^\alpha = C$.*

Upon Theorem (3.6.1), we can study the behavior of $g(t|\alpha)$ in a neighbourhood of parameters as $t \to 0^+$. Recall that $\log(s\hat{g}(s|\alpha)) \sim 2\sqrt{\alpha_1}(\sqrt{\alpha_3} - \sqrt{\alpha_4})s^{\frac{1}{2}}$, as $s \to +\infty$, then from the De Bruijn's Tauberian theorem, for fixed α, we have:

$$\log g(t|\alpha) \sim -\alpha_1(\sqrt{\alpha_3} - \sqrt{\alpha_4})^2 \frac{1}{t} \quad as \quad t \to 0^+. \tag{3.6.3}$$

Thus we can write $g(t|\alpha) = \exp(-\frac{\alpha_1(\sqrt{\alpha_3}-\sqrt{\alpha_4})^2}{t} + \frac{l(t|\alpha)}{t})$. Since we only know for fixed α, $l(t|\alpha) \to 0$, we would like to extend it to a neighbourhood argument:

Lemma 3.6.5. $\forall$ α^0 *is in the parameter space, there exits a closure of α^0, such that $l(t|\alpha)$ goes to zero uniformly as $t \to 0^+$.*

3.7 SOME PARTIAL RESULTS AND CONJECTURES OF THE MLE THEOREM

We have explored some tail behaviors of the density function $g(t|\alpha)$ as $t \to +\infty$ and $t \to 0^+$, now we can discuss the MLE theorem. As we know, the MLE result consists of two parts: consistency and asymptotic normality. Since in most literature, and in order to make all the arguments rigorous, we do need to constrain the parameter space to be a compact set $\mathcal{K}$ of $\mathcal{A} = \{(\alpha_1, \alpha_2, \alpha_3, \alpha_4) : \alpha_i > 0\}$. There is some compactification technique to handle this issue, for example the Cauchy likelihood case in ($Van\ derVaart$, 2000), but the investigation of $\log g(t|\alpha)$ as $\alpha_i \to 0^+$ and $\alpha_3 \to \alpha_4$ is really difficult. While in practice, researchers will usually have the knowledge of determining a smaller range of the parameters rather than $\mathcal{A}$, it is not a big loss to assume that the parameter space is $\mathcal{K}$ instead of $\mathcal{A}$, for example, let $\mathcal{K}$ be $\{(\alpha_1, \alpha_2, \alpha_3, \alpha_4) : 0 < \epsilon \leq \alpha_i \leq M, \alpha_3 + \epsilon \leq \alpha_4\}$, with ϵ close to 0 and M very large. Let's first explore the property of consistency:

Lemma 3.7.1. *If the parameter space is $\mathcal{K}$, then the any MLE $\hat{\alpha}_n$ for α is consistent.*

However, when it comes to the asymptotic normality, we encounter some serious challenges. According to (Lehmann and Casella, 2006), the regularity conditions include the differentiability and its bounded property of partial derivatives up to the third order of the density functions, which is too strong. (Van der Vaart, 2000) states a relatively relaxing condition: differentiability of the root density $\alpha \to \sqrt{g(t|\alpha)}$ in quadratic mean, which entails the existence of a vector of measurable functions $\dot{s_\alpha}$ such that:

$$\int_0^\infty (\sqrt{g(t|\alpha + h)} - \sqrt{g(t|\alpha)} - \frac{1}{2} h^T \dot{s_\alpha}(t))^2 dt = o(||h||^2) \tag{3.7.1}$$

along with a Lipschitz condition: there exists a measurable function $\dot{m}(t)$ with finite second moment under probability measure with density function $g(t|\alpha^0)$ and $\forall \alpha\ \beta$ in a neighbourhood of α^0 satisfying:

$$|\log g(t|\alpha) - \log g(t|\beta)| \leq \dot{m}(t)||\alpha - \beta|| \tag{3.7.2}$$

With only accessibility to the Laplace transform of $g(t|\alpha)$, from Lemma 3.5.9, we can observe that when t is bounded away from zero, the two conditions can be satisfied. But for $t \to 0^+$, by Equation (3.6.4), $\frac{\partial \log g(t|\alpha)}{\partial \alpha} = (-\frac{\partial\{\alpha_1(\sqrt{\alpha_3 - \alpha_4})^2\}}{\partial \alpha} + \frac{\partial l(t|\alpha)}{\partial \alpha})/t$, at this moment, we have no controls for $\frac{\partial l(t|\alpha)}{\partial \alpha}$. Unlike $g(t|\alpha)$, we can not regard $\frac{\partial g(t|\alpha)}{\partial \alpha_i}$ as measure when t is near zero, which means that we can not apply Theorem 3.6.1. However, because $l(t|\alpha)$ goes to 0 uniformly, it is no that insane to have our conjecture:

Assumption 1. *In a closed neighbourhood of α^0, when $0 < t < \delta$, $\frac{\partial l(t|\alpha)}{\partial \alpha}$ can be bounded by some polynomials in $\frac{1}{t}$.*

There is some discussion about the alternative assumptions of an MLE, for example in (LeCam, 1970), the author listed some alternative conditions. While we can see that the condition of differentiability in quadratic mean is crucial, and it also requires some other conditions for uniform integrability behavior in a neighbourhood of the true parameter, see Assumption A1 and A2 of (LeCam, 1970).

Suppose that the true value of $\alpha^0 = (\alpha_1^0, \alpha_2^0, \alpha_3^0, \alpha_4^0)$ is in the interior of a compact set $\mathcal{K}$, which is a subset of $\mathcal{A} = \{(\alpha_1, \alpha_2, \alpha_3, \alpha_4), \alpha_i > 0\}$ and Assumption 1 is valid. Given only first passage time observations t_1, t_2,..., t_n, there exits an MLE $\hat{\alpha}_n$ for α^0 satisfying: $\sqrt{n}(\hat{\alpha}_n - \alpha^0)$ is asymptotically normal with mean 0 and covariance matrix $I(\alpha^0)^{-1}$.

From the discussion above, we find that the difficulty is when $t \to 0^+$, so we propose a conditional version of MLE. By conditioning on the event that the first passage time is greater than $\Delta > 0$, we can obtain a new class of conditional density functions:

$$\tilde{g}(t|\alpha, \Delta) = \frac{g(t|\alpha)}{1 - \int_0^\Delta g(l|\alpha) dl} \quad \text{for} \quad t \geq \Delta t \tag{3.7.3}$$

Instead of dealing with $g(t|\alpha)$, if we use $\tilde{g}(t|\alpha, \Delta)$ as our density function, we will have both consistency and asymptotic normality.

Theorem 3.7.2. *Assume the true value of $\alpha^0 = (\alpha_1^0, \alpha_2^0, \alpha_3^0, \alpha_4^0)$ is in the interior of $\mathcal{K}$, given first passage time observations t_1, t_2,..., t_n with density function $\tilde{g}(t|\alpha, \Delta)$, if the information matrix $\tilde{I}(\alpha_0)$ is invertible, then there exists an MLE $\hat{\hat{\alpha}}_n$ such that:*

$$\sqrt{n}(\hat{\hat{\alpha}}_n - \alpha^0) \xrightarrow{d} N(0, \tilde{I}(\alpha_0)^{-1}) \tag{3.7.4}$$

Note that the difference between $\hat{\alpha}_n$ and $\hat{\hat{\alpha}}_n$ can be small, since we can predetermine some really tiny Δ. When Δ is extremely small, numerically we would expect $\hat{\alpha}_n \approx \hat{\hat{\alpha}}_n$. Then in the simulation study, although we rely on Theorem 3.7.2, we deal with the unconditional case directly.

3.8 SIMULATION STUDY (ONGOING)

After all these studies of the MLE theorem based on first passage times of the CIR process to a constant boundary, we have already built up our theoretical framework for the simulation study. The idea is straightforward, and we can proceed with the following steps:

(i) Simulate first passage times for a CIR process crossing up (or down) a constant boundary with some predetermined parameters:

Because we do not have access to the density function directly, in order to simulate the first passage times, we have to simulate the diffusion process path by path. Here we apply the method described in (Giraudo et al., 2001). The paper proposed a Monte Carlo method for the evaluation of first passage times of diffusion processes through boundaries, which accounted for undetected crossings that may occur inside each discretization interval of the simulated diffusion processes. We outline the procedure:

Given a set of α, β, k, y, y_c as defined in Equation (3.1.1) and (3.1.2). Simulate a discrete process by an iterative scheme: let

$$\tau_n = nh, \quad n = 0, 1, 2, \cdots, \frac{T}{h} \tag{3.8.1}$$

be a partition of the time interval $[0, T]$. Then construct Y_n using

$$\begin{aligned} Y_0 &= y \\ Y_{n+1} &= Y_n + \Phi(Y_n, h, \Delta W_{n+1}, \Delta Z_{n+1}), \quad n = 0, 1, 2, \cdots, \frac{T}{n} - 1 \end{aligned} \tag{3.8.2}$$

Where ΔW_n are independent $N(0, h^2)$, ΔZ_n are independent $N(0, \frac{h^2}{3})$, with $E(\Delta Z_n \Delta W_n) = \frac{1}{2}h^2$, and

$$\begin{aligned} \Phi(Y_n, h, \Delta W_{n+1}, \Delta Z_{n+1}) = {} & (-\alpha Y_n + \beta)h + k\sqrt{Y_n}\Delta W_{n+1} + \frac{1}{4}k^2(\Delta W_{n+1}^2 - h) \\ & - \alpha k\sqrt{Y_n}\Delta Z_{n+1} + \frac{1}{2}\alpha(\alpha Y_n - \beta)h^2 \\ & + [\frac{1}{2}\frac{k}{\sqrt{Y_n}}(-\alpha Y_n + \beta) - \frac{1}{8}k^2\sqrt{Y_n}](h\Delta W_{n+1} - \Delta Z_{n+1}) \end{aligned}$$

The equation above arises from the analysis in (Talay, 1994), which is a better approximation of the diffusion process than regular Eular scheme. Apply Equation (3.8.2) iteratively until we find some $Y_n \geq y_c$. In order to account for the passage between two adjacent time points, we make use of a simulation technique that involves the following tied-down processes to evaluate the crossing probabilities between two adjacent points. Since we have fixed endpoints Y_n and Y_{n+1} at time τ_n and τ_{n+1}, we have to simulate a bridge process constrained between Y_n and Y_{n+1}. Then the stochastic differential equation associated with the bridge process is:

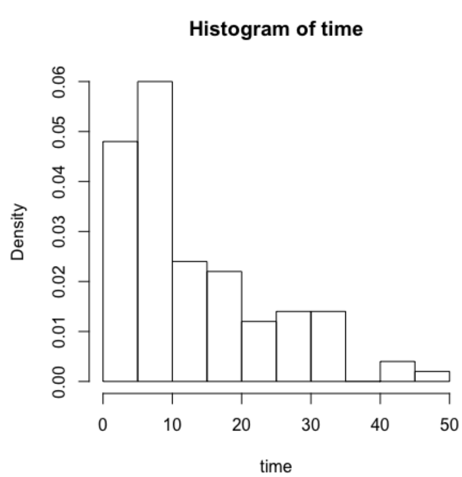
$$d\hat{Y}^{[a,b]}(t) = \mu^{[a,b]}(\hat{Y}^{[a,b]}(t), t) + \sigma^{[a,b]}(\hat{Y}^{[a,b]}(t), t)dB_t \tag{3.8.3}$$

with $a = \tau_n$, $b = \tau_{n+1}$, $\hat{Y}^{[a,b]}(a) = Y_n$, $\hat{Y}^{[a,b]}(b) = Y_{n+1}$ and

$$\begin{aligned}\mu^{[a,b]}(x,t) =& \beta + x[-\alpha + 2\frac{\alpha\exp(-\alpha(b-t))}{\exp(-\alpha(b-t))-1}] \\ &- \frac{2\alpha\sqrt{x\hat{Y}^{[a,b]}(b)\exp(-\alpha(b-t))}}{\exp(-\alpha(b-t))-1}\frac{I_{\frac{2\beta}{k^2}}[\frac{2\alpha\sqrt{x\hat{Y}^{[a,b]}(b)\exp(-\alpha(b-t))}}{r(\exp(-\alpha(b-t))-1)}]}{I_{\frac{2\beta}{k^2}-1}[\frac{2\alpha\sqrt{x\hat{Y}^{[a,b]}(b)\exp(-\alpha(b-t))}}{r(\exp(-\alpha(b-t))-1)}]} \\ \sigma^{[a,b]}(x,t) =& k\sqrt{x}\end{aligned} \tag{3.8.4}$$

We simulate the process between each Y_n and Y_{n+1} M times, use the number of times that there exists a point that exceeds y_c over M to approximate the probability of the occurrence of passage between τ_n and τ_{n+1}. Compare this probability with a number generated from a uniform(0, 1) distribution, if the probability is larger, we conclude that there is a passage between τ_n and τ_{n+1}, and assume it happens at $\frac{\tau_n+\tau_{n+1}}{2}$. The following figure is a histogram of 100 simulated first passage times with parameters $\alpha = 0.2$, $\beta = 2$, $k = 1$, $y = 5$ and $y_c = 15$.

(ii) Search for a good way to numerically invert the Laplace transforms of the density functions



and the derivative of density functions:

By Equation (3.4.3) and (3.4.4), we know that the Bromwich integral is valid for us to access the density function and its derivatives. There are some classical Laplace inversion methods in (Cohen, 2007). We need one that can evaluate the density function efficiently for different time points, so we applied the method proposed in (De Hoog et al., 1982); define:

$$g^{[\infty]}(t|\alpha) = \frac{1}{T}\exp(\gamma t)[\frac{\hat{g}(\gamma|\alpha)}{2} + \sum_{k=1}^{\infty}\Re\{\hat{g}(\gamma + \frac{ik\pi}{T})\exp(\frac{ik\pi t}{T})\}] \tag{3.8.5}$$

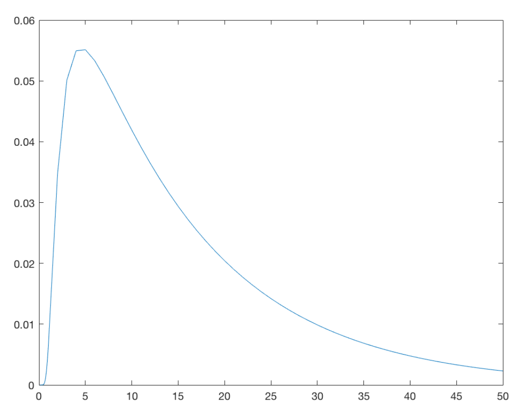
By Equation (11) in (De Hoog et al., 1982),

$$g^{[\infty]}(t|\alpha) = g(t|\alpha) + \sum_{k=1}^{\infty}\exp(-2\gamma kT)\hat{g}(2kT + t|\alpha) \tag{3.8.6}$$

we are able to control the difference between $g^{[\infty]}(t|\alpha)$ and $g(t|\alpha)$ through $2\gamma T$. So we can use

$$g^{[N]}(t|\alpha) = \frac{1}{T}\exp(\gamma t)[\frac{\hat{g}(\gamma|\alpha)}{2} + \sum_{k=1}^{N}\Re\{\hat{g}(\gamma + \frac{ik\pi}{T}|\alpha)\exp(\frac{ik\pi t}{T})\}] \tag{3.8.7}$$

to estimate $g(t|\alpha)$. From Equation (3.8.7), we find that for different t, we only need to calculate the coefficients $\hat{g}(\gamma + \frac{ik\pi}{T}|\alpha)$ once, which is more effective than other methods. However, if we require the truncation error $\frac{1}{T}\exp(\gamma t)\sum_{k=N+1}^{\infty}\Re\{\hat{g}(\gamma + \frac{ik\pi}{T})\exp(\frac{ik\pi t}{T})\}$ to be small (10^{-7}), using Equation (3.8.7) directly asks for more than 4000 terms typically, which is really time inefficient. So we used Q-D algorithm described in (De Hoog et al., 1982) to accelerate the computation. In this way, we only need to calculate around 100 terms. The following figure is the estimated density function of the first passage times with $\alpha_1 = 5$, $\alpha_2 = 4$, $\alpha_3 = 2$ and $\alpha_4 = 6$. The shape matches our observation of the histogram previously. The calculation for the density function itself



is straightforward and fast, but when it comes to the evaluation of derivatives, it is less stable and more time consuming, we are still trying to come up with a more efficient solution.

(iii) Use the Newton Raphson method to obtain the MLE, plug in the estimates to get estimates for the information matrix and construct confidence interval for the unknown parameters.

4.0 STATISTICAL INFERENCE FOR THE FIRST PASSAGE TIME PROBLEM OF GENERAL ONE-DIMENSIONAL DIFFUSION PROCESSES

In Chapter 5, we investigated the inference problem for the first passage time of the CIR model, and noticed that its Laplace transform $\hat{g}(s|\alpha)$ (for up crossing) and $\hat{h}(s|\alpha)$ (for down crossing) both decay exponentially in a rate of some constant multiplied by $\sqrt{|s|}$, which coincides with the case of the OU process and the ROU process. Meanwhile, the density functions are all decreasing exponentially in a rate of some constant multiplied by t. These observations make us think about generalizing the procedure as we did in Chapter 3 to some other process, even a general time homogeneous diffusion process with the SDE: $dX(t) = \mu(X(t), \theta)dX(t) + \sigma(X(t), \theta)dB(t)$. As the tractability of the Laplace transform of the first passage time depends on the resolvability of some ODE or PDE, this method has its limitation. However, we can try the integral equation method discussed in section 2.1.4. It can be really technical since we need to study the uniqueness property of solutions to some integral equations, along with their derivatives. We also need to think carefully whether the MLE theorem is available for us to do statistical inference, if not, what other techniques we can rely on. We may try some other methods and all these are left as future work.

5.0 FIRST PASSAGE TIME PROBLEMS FOR SOME TWO-DIMENSIONAL DIFFUSION PROCESSES

The first passage time problem for two-dimensional diffusion processes is very challenging, for an infinite wedge case, even the Brownian motion with drift has an extremely complicated form of Laplace transform for its first passage time (Equation 2.2.9). In this chapter, we discuss several special processes that are feasible for us to make statistical inference based on its boundary crossing time. Thus, we only present partial work, and leave it for future research.

5.1 FIRST PASSAGE TIME PROBLEM FOR A TWO-DIMENSIONAL OU PROCESS

We start with a discussion of a very special two-dimensional OU processes with the following stochastic differential equation:

$$dX(t) = -X(t)dt + \Sigma^{\frac{1}{2}} dB(t) \tag{5.1.1}$$

where $X(t) = \begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix}$, $X(0) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $\Sigma = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$, $\Sigma^{\frac{1}{2}} = \begin{pmatrix} \cos\beta & \sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}$, $\rho = \sin 2\beta$, $-\frac{\pi}{4} < \beta < -\frac{\pi}{4}$.

Define the first passage times :

$$\tau_1 = \inf\{t : X_1(t) = a_1\}, \qquad \tau_2 = \inf\{t : X_2(t) = a_2\}, \qquad \tau = \tau_1 \wedge \tau_2$$

By linear transformation:

$$\begin{pmatrix} Y_1(t) \\ Y_2(t) \end{pmatrix} = \Sigma^{-\frac{1}{2}} \begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix}$$

We have:

$$d\begin{pmatrix} Y_1(t) \\ Y_2(t) \end{pmatrix} = -\begin{pmatrix} Y_1(t) \\ Y_2(t) \end{pmatrix} + \begin{pmatrix} dB_1(t) \\ dB_2(t) \end{pmatrix}$$

and the boundary is mapped to a wedge:

$$\begin{cases} x_1 \cos\beta + x_2 \sin\beta = a_1 \cos 2\beta \\ x_1 \sin\beta + x_2 \cos\beta = a_2 \cos 2\beta \end{cases}$$

We are interested in the Laplace transform of τ

$$E_x[\exp(c\tau)]$$

Lemma 5.1.1. *If there is a bounded solution for the following pde with the boundary condition, it must be* $E_x[\exp(c\tau)]$*:*

$$\frac{1}{2}\Delta f + cf - \nabla f \cdot Y = 0 \tag{5.1.2}$$

$$f \equiv 1 \text{ on } \partial G. \tag{5.1.3}$$

where f is continuous in G (G is generally an open set in R^2; here G is the wedge).

Now we are focused on the second order PDE

$$\frac{1}{2}\frac{\partial^2 f}{\partial x_1^2} + \frac{1}{2}\frac{\partial^2 f}{\partial x_2^2} - x_1\frac{\partial f}{\partial x_1} - x_2\frac{\partial f}{\partial x_2} + cf = 0 \tag{5.1.4}$$

with a boundary condition that $f \equiv 1$ on ∂G. After a shift and rotation, the PDE can be written as

$$\frac{1}{2}\frac{\partial^2 f}{\partial x_1^2} + \frac{1}{2}\frac{\partial^2 f}{\partial x_2^2} - x_1\frac{\partial f}{\partial x_1} - x_2\frac{\partial f}{\partial x_2} - \mu_1\frac{\partial f}{\partial x_1} - \mu_2\frac{\partial f}{\partial x_2} + cf = 0 \tag{5.1.5}$$

where $\mu_1 = a_1 \cos\beta - a_2 \sin\beta$ and $\mu_2 = a_2 \cos\beta - a_1 \sin\beta$. The area G became a wedge on x_1 axis with an angle γ where $\cos\gamma = |\rho|$, and the peak of the wedge is at the origin. $f \equiv 1$ on ∂G. For even more simplicity, consider a special case such that $a_1 = 0$, $a_2 = 0$, then we need to solve:

$$\frac{1}{2}\frac{\partial^2 f}{\partial x_1^2} + \frac{1}{2}\frac{\partial^2 f}{\partial x_2^2} - x_1\frac{\partial f}{\partial x_1} - x_2\frac{\partial f}{\partial x_2} + cf = 0 \tag{5.1.6}$$

Let $k = f - 1$, then

$$\frac{1}{2}\frac{\partial^2 k}{\partial x_1^2} + \frac{1}{2}\frac{\partial^2 k}{\partial x_2^2} - x_1\frac{\partial k}{\partial x_1} - x_2\frac{\partial k}{\partial x_2} + ck = -c \tag{5.1.7}$$

with boundary condition $k = 0$ on ∂G.

By polar coordinate transformation, $x_1 = r\cos\theta$, $x_2 = r\sin\theta$, we have the following form:

$$\frac{1}{2}\left(\frac{\partial^2 k}{\partial r^2} + \frac{1}{r}\frac{\partial k}{\partial r} + \frac{1}{r^2}\frac{\partial^2 k}{\partial \theta^2}\right) - r\frac{\partial k}{\partial r} + ck = -c \tag{5.1.8}$$

First we take the finite Fourier transform $W_n(r) = \int_0^\gamma \sqrt{\frac{2}{\gamma}}\sin(v_n\eta)k(r,\eta)d\eta$, where $v_n = \frac{n\pi}{\gamma}$,and the following PDE arises.

$$\frac{1}{2}\left(\frac{\partial W_n^2}{\partial r^2} + \frac{1}{r}\frac{\partial W_n}{\partial r} - \frac{v_n^2}{r^2}W_n\right) - \frac{v_n}{r^2}\sqrt{\frac{2}{\gamma}}(k(r,\gamma) - k(r,0)) - r\frac{\partial W_n}{\partial r} + cW_n = \int_0^\gamma \sqrt{\frac{2}{\gamma}}\sin(v_n\eta)(-c)d\eta$$

Thus

$$\frac{1}{2}\left(\frac{\partial W_n^2}{\partial r^2} + \frac{1}{r}\frac{\partial W_n}{\partial r} - \frac{v_n^2}{r^2}W_n\right) - r\frac{\partial W_n}{\partial r} + cW_n = c\sqrt{\frac{2}{\gamma}}\frac{\cos(n\pi) - 1}{v_n} \tag{5.1.9}$$

For the homogeneous case:

$$\frac{1}{2}\left(\frac{\partial W_n^2}{\partial r^2} + \frac{1}{r}\frac{\partial W_n}{\partial r} - \frac{v_n^2}{r^2}W_n\right) - r\frac{\partial W_n}{\partial r} + cW_n = 0$$

One of our candidates for its solution is $W_n(r) = r^{v_n}g(r^2)$, then $g(r^2)$should satisfy:

$$r^2 g''(r^2) + (v_n + 1 - r^2)g'(r^2) + \left(\frac{c}{2} - \frac{v_n}{2}\right)g(r^2) = 0 \tag{5.1.10}$$

From which we know two linearly independent solutions for the homogeneous case are $r^{v_n}F(\frac{v_n}{2} - \frac{c}{2}, 1 + v_n, r^2)$ and $r^{v_n}U(\frac{v_n}{2} - \frac{c}{2}, 1 + v_n, r^2)$, where F and U are confluent hypergeometric functions as introduced in earlier chapters. Then with the Wronskian theorem, general solutions for the non-homogeneous case are

$$W_n(r) = a_1 r^{v_n}F\left(\frac{v_n}{2} - \frac{c}{2}, 1 + v_n, r^2\right) + a_2 r^{v_n}U\left(\frac{v_n}{2} - \frac{c}{2}, 1 + v_n, r^2\right) + T(r) \tag{5.1.11}$$

$$\begin{aligned} T_r = {} & r_n^{v_n}U_n(r^2)\int_0^r l^{v_n+1}F_n(l^2)c\sqrt{\frac{2}{\gamma}}\frac{\cos(n\pi) - 1}{v_n}\left(-\frac{1}{2}\right)\frac{\Gamma(\frac{v_n}{2} - \frac{c}{2})}{\Gamma(1 + v_n)}\exp(-l^2)dl \\ & + r^{v_n}F_n(r^2)\int_r^\infty l^{v_n+1}U_n(l^2)c\sqrt{\frac{2}{\gamma}}\frac{\cos(n\pi) - 1}{v_n}\left(-\frac{1}{2}\right)\frac{\Gamma(\frac{v_n}{2} - \frac{c}{2})}{\Gamma(1 + v_n)}\exp(-l^2)dl \end{aligned} \tag{5.1.12}$$

where $F_n(r^2) = F(\frac{v_n}{2} - \frac{c}{2}, 1 + v_n$ and $U_n(r^2) = U(\frac{v_n}{2} - \frac{c}{2}, 1 + v_n, r^2)$.

Next, we come to show that if there is a bounded solution on G for Equation (5.1.11) and (5.1.12), then we must have $W_n(r) = T(r)$. If $f(r, \theta)$ is bounded, then $k(r, \theta)$ is also bounded. Therefore, $W_n(r)$ is bounded for $0 < r < \infty$. While $U(\frac{v_n}{2} - \frac{c}{2}, 1 + v_n, r^2) \simeq A\exp(r^2)r^{-c-v_n-2}$, we know $r^{v_n}F(\frac{v_n}{2} - \frac{c}{2}, 1 + v_n, r^2) \to \infty$ as $r \to \infty$. Similarly, $U(\frac{v_n}{2} - \frac{c}{2}, 1 + v_n, r^2) \simeq Br^{-2v_n}$, then $r^{v_n}U(\frac{v_n}{2} - \frac{c}{2}, 1 + v_n) \to \infty$ as $r \to 0$. Hence, we must have $a_1 = a_2 = 0$. Then we use inversion formula to obtain our candidate for $k(r, \theta)$ and $f(r, \theta) = k(r, \theta) + 1$:

$$k(r, \theta) = \sum_{n=1}^{\infty} \sqrt{\frac{2}{\gamma}} \sin(v_n\theta) W_n(r) \tag{5.1.13}$$

From the above analysis, we can see that even for this simplest case, with the drift part being $-X(t)dt$ and only one unknown parameter ρ in the diffusion part, possible candidate of the Laplace transform for the first passage time of $X(t)$ through the horizontal axis and the vertical axis can still be very involved. To guarantee that $f(r, \theta)$ is the Laplace transform of τ, we also need to verify some uniform convergent conditions since we have an infinite sum. However, for this toy problem, we can solve it using another technique.

Consider $dX(t) = -X(t)dt + \Sigma^{\frac{1}{2}}dB(t)$, where $X(0) = x_0$, Q is some constant matrix and Σ has the same form as described in Equation (5.1.1). Then the stochastic equation has a strong solution:

$$X(t) = \exp(-tQ)x_0 + \int_0^t \exp(-(t-s)Q)\Sigma^{\frac{1}{2}}dB(s) \tag{5.1.14}$$

With Q being identity matrix, define $Y(t) = \int_0^t \exp(sI)\Sigma^{\frac{1}{2}}dB(s)$ and $c_t = \inf\{s : \frac{1}{2}(\exp(2s) - 1) > t\} = \frac{1}{2}\ln(2t+1)$, then by the time change technique, we know $Y(c_t)$ has the same distribution with $\Sigma^{\frac{1}{2}}B(t)$. On the other hand, c_τ has the same distribution with T, which is the first passage time of $\Sigma^{\frac{1}{2}}B(t)$ hitting either the horizontal axis or the vertical axis. The distribution of T has been discussed in Section 2.2.1. However, because of the extra multiplier $\exp(-t)$, this time change technique does not work for the general constant boundary case.

The difficulty of the wedge boundary crossing problem intrigues us to think about other types of boundaries. Sometimes, when the boundary is a circle, the Laplace transform of the first passage time for a two-dimensional diffusion process can be analytically tractable. This work is still undergoing.

APPENDIX A

PROOF IN SECTION 3.2

A.1 PROOF OF LEMMA 3.2.1

Proof. When $\Re\mu \geq 0$, t is real (we can make it positive), following the notation of Volkmer (Volkmer et al., 2016) (page 11), write $W_3(te^{i\theta}, \mu, e^{-i\theta}x) = e^{i\theta}\tilde{W}_3(t, \mu, x)$, where W_3 is a solution to the ODE:

$$w'' = \frac{1}{z}w'(z) + (u^2 + \frac{\mu^2 - 1}{z^2} + f(z))w(z).$$

$\tilde{W}_3$ is a solution to the ODE

$$\tilde{w}'' = \frac{1}{x}\tilde{w}'(x) + \left[t^2 + \frac{\mu^2 - 1}{x^2} + e^{-2i\theta}f(e^{-i\theta}x)\right]\tilde{w}(x).$$

Because t is real in $\tilde{W}_3$, we can apply Olver's work (Olver, 1956), which gives us an expansion for $\tilde{W}_3$:

$$\tilde{W}_3(t, \mu, x) = xI_\mu(tx)\left[\sum_{s=0}^{N-1}\frac{\tilde{A}_s(x)}{t^{2s}} + g_2(t, x)\right] + \frac{x}{t}I_{\mu+1}(\sum_{s=0}^{N-1}\frac{\tilde{B}_s(x)}{t^{2s}} + xh_2(t, x)) \qquad \text{(A.1.1)}$$

where $|g_2| + |h_2| \leq \frac{K_1}{t^{2N}}$ for $0 < |x| \leq R$, $t \geq t_1$. and K_1 is a positive constant depending on R, N, μ and t_1. Next, let $z = e^{-i\theta}x$, $u = te^{i\theta}$; then we have

$$W_3(te^{i\theta}, \mu, z) = zI_\mu(uz)(\sum_{s=0}^{N-1}\frac{\tilde{A}_s(e^{i\theta}z)}{t^{2s}} + g_2(t, e^{i\theta}z)) + \frac{z}{t}I_{\mu+1}(\sum_{s=0}^{N-1}\frac{\tilde{B}_s(e^{i\theta}z)}{t^{2s}} + e^{i\theta}zh_2(t, e^{i\theta}z)).$$

On the other hand, $F(\frac{1}{4}u^2+\frac{1}{2}b,b,z^2)=\frac{\Gamma(b)}{2^{1-b}u^{b-1}}e^{\frac{1}{2}z^2}z^{-b}W_3(u,\mu,z)$ with $\mu=b-1$ and $f=z^2$, so we have proved the case for $\Re b\geq 1$. When $0<\Re b<1$, we can not use Olver's work directly, as $\Re\mu=\Re b-1<0$. However, in the proof of Theorem 7.1 in (Volkmer et al., 2016), the author showed that Equation (A.1.1) is still valid. Let $z=e^{-i\theta}x$, $u=te^{i\theta}$ again, the case for $0<\Re b<1$ is proved. □

A.2 PROOF OF LEMMA 3.2.4

Proof. Here fix R and b. By Lemma 3.2.1, 3.2.3, there is a constant $C_1>0$ such that for sufficiently large $|u|$, we have:

$$|\beta_2(u)e^{-z^2/2}z^bU(a,b,c)|\leq C_1|u^{1-b}|Q \tag{A.2.1}$$

With Lemma 3.2.1 and the asymptotic expansion of the modified Bessel function $I_{b-1}(uz)$ in (Olver, 2010), we have that for some constant $C_2>0$,

$$|e^{-z^2/2}z^bF(a,b,c)|\geq C_2|u^{\frac{1}{2}-b}||e^{zu}| \tag{A.2.2}$$

In the same way, with the asymptotic expansion of the modified Bessel function $K_{b-1}(uz)$ and Equation (3.2.3), we have for some constant $C_3>0$,

$$|e^{-i\theta}W_2(t,\mu,x)|\leq C_3|u^{-\frac{1}{2}}|e^{-zu}| \tag{A.2.3}$$

So we must have

$$\begin{aligned}|\beta_1(u)| &\leq \frac{C_3}{C_2}|u^{b-1}||e^{-2zu}|+\frac{C_1Q}{C_2}|u^{\frac{1}{2}}||e^{-zu}| \\ &= \frac{C_3}{C_2}t^{b-1}e^{-2zt\cos\theta}+\frac{C_1Q}{C_2}t^{\frac{1}{2}}e^{-zt\cos\theta} \\ &\leq \frac{C_3}{C_2}t^{b-1}e^{-2zt\cos(\frac{\pi}{2}-\delta)}+\frac{C_1Q}{C_2}t^{\frac{1}{2}}e^{-zt\cos(\frac{\pi}{2}-\delta)}\end{aligned} \tag{A.2.4}$$

As $|u|$ goes to ∞, we must have $t\to\infty$. Take $z=R$, we know $\beta_1(u)=\mathcal{O}(e^{-tq\cos(\frac{\pi}{2}-\delta)})$ □

A.3 PROOF OF LEMMA 3.2.5

Proof. Let $M(u,z) = \beta_1(u)e^{-z^2/2}z^bF(a,b,z^2)$. By Lemma 3.2.1 and Lemma 3.2.4, we know that

$$|M(u,z)| \leq C_4 e^{-tq\cos(\frac{\pi}{2}-\delta)}|z|(|I_{b-1}(uz)| + \frac{1}{t}|I_b(uz)|) \tag{A.3.1}$$

On the other hand, $|I_v(uz)| \leq C_5 e^{|uz|}$ for $|\arg(uz)| = |\theta| \leq \frac{3\pi}{2}$, $\Re(v) \geq 0$. So we have

$$|M(u,z)| \leq C_6|z|e^{tz-tq\cos(\frac{\pi}{2}-\delta)} \tag{A.3.2}$$

when $0 < z \leq \frac{R\cos(\frac{\pi}{2}-\delta)}{3}$, $t > t_1$.

Meanwhile, $M(u,z) = zK_{b-1}(uz)g(u,z) - \frac{u}{z}K_b(uz)zh(u,z)$ with $g(u,z) = uI_b(uz)M(u,z)$ and $h(u,z) = -\frac{u^2}{z}I_{b-1}(uz)M(u,z)$. Along with Equation (A.3.2), we know that when $0 < z \leq R\cos(\frac{\pi}{2}-\delta)/3$,

$$\begin{aligned} |g(u,z)| &\leq C_5C_6\frac{R\cos(\frac{\pi}{2}-\delta)}{3}|t|\exp\left[t\frac{2R\cos(\frac{\pi}{2}-\delta)}{3} - tq\cos(\frac{\pi}{2}-\delta)\right] \\ |h(u,z)| &\leq C_5C_6t^2\exp\left[t\frac{2R\cos(\frac{\pi}{2}-\delta)}{3} - tq\cos(\frac{\pi}{2}-\delta)\right] \end{aligned} \tag{A.3.3}$$

Let $q = \frac{3R}{4}$, and recall that $e^{-i\theta}\tilde{W}_2(t,\mu,x) = \beta_1(u)e^{-\frac{1}{2}z^2}z^bF(a,b,z^2) + \beta_2(u)e^{-\frac{1}{2}z^2}z^bU(a,b,z^2)$ has the expansion

$$e^{-i\theta}\tilde{W}_2(t,\mu,x) = zK_\mu(uz)(\sum_{s=0}^{N-1}\frac{\tilde{A}_s(e^{i\theta}z)}{t^{2s}} + g_3(t,e^{i\theta}z))$$

$$-\frac{z}{t}K_{\mu+1}(uz)(\sum_{s=0}^{N-1}\frac{\tilde{B}_s(e^{i\theta}z)}{t^{2s}} + uh_3(t,e^{i\theta}z)).$$

Define $g_4(t,\theta,z) = g_3(t,e^{i\theta}z) - g(u,z)$, $h_4(t,\theta,z) = h_3(t,e^{i\theta}z) - g(u,z)$, to get the desired expansion. □

A.4 PROOF OF LEMMA 3.2.6

Proof. Set $N(u,z) = \beta_2(u)e^{-z^2/2}z^b U(a,b,z^2)$, by Equation (13.2.12) in (Olver, 2010):

$$U(a,b,(ze^{i\pi})^2) = e^{-2\pi ib}U(a,b,z^2) + \frac{2\pi i e^{-\pi ib}}{\Gamma(b)\Gamma(1+a-b)}F(a,b,z^2)$$

along with the relationship of $F(a,b,z^2)$ and W_3 we obtain

$$N(u,ze^{i\pi}) - e^{-i\pi b}N(u,z) = \beta_2(u)\frac{\pi i 2^b u^{1-b}}{\Gamma(1+a-b)}W_3(u,b-1,z) \tag{A.4.1}$$

Next, we expand the two terms on the left hand side. Because $e^{i\pi} = -1$, we know $ze^{i\pi}$ is still real. We can apply Lemma 3.2.5 to both $N(u,ze^{i\pi})$ and $N(u,z)$.

$$\begin{aligned} N(u,ze^{i\pi}) &= ze^{i\pi}K_{b-1}(ze^{i\pi}u)\left[\sum_{s=0}^{N-1}\frac{\tilde{A}_s(e^{i\theta}ze^{i\pi})}{t^{2s}} + g_4(t,\theta,ze^{i\pi})\right] \\ &\quad - \frac{ze^{i\pi}}{t}K_b(ze^{i\pi}u)\left[\sum_{s=0}^{N-1}\frac{\tilde{B}_s(e^{i\theta}ze^{i\pi})}{t^{2s}} + ze^{i\pi}h_4(t,\theta,ze^{i\pi})\right] \end{aligned} \tag{A.4.2}$$

Notice that $\tilde{A}_s(e^{i\theta}ze^{i\pi})$ are even functions of z, $\tilde{B}_s(e^{i\theta}ze^{i\pi})$ are odd functions in z (both can be derived by induction), and note that

$$K_b(ze^{i\pi}) = e^{-i\pi b}K_b(z) - i\pi I_b(z)$$

Equation A(4.2) can be written as

$$\begin{aligned} N(u,ze^{i\pi}) &= -z(e^{-i\pi b}K_{b-1}(zu) - i\pi I_{b-1}(zu))\left[\sum_{s=0}^{N-1}\frac{\tilde{A}_s(e^{i\theta}z)}{t^{2s}} + g_4(t,\theta,ze^{i\pi})\right] \\ &\quad - \frac{z}{t}(e^{-i\pi b}K_b(zu) - i\pi I_b(zu))\left[\sum_{s=0}^{N-1}\frac{\tilde{B}_s(e^{i\theta}z)}{t^{2s}} + ze^{i\pi}h_4(t,\theta,ze^{i\pi})\right] \end{aligned} \tag{A.4.3}$$

Apply Lemma 3.2.5 to $N(u,z)$ and expand $W_3(u,b-1,z)$, to get

$$\begin{aligned}
N(u,ze^{i\pi})-e^{-i\pi b}N(u,z)=&-ze^{-i\pi b}K_{b-1}(zu)\Big[2\sum_{s=0}^{N-1}\frac{\tilde{A}_s(e^{i\theta}z)}{t^{2s}}+g_4(t,\theta,ze^{i\pi})+g_4(t,\theta,z)\\
&-\frac{z}{t}e^{-i\pi b}K_b(zu)(ze^{i\pi}h_4(t,\theta,ze^{i\pi})-zh_4(t,\theta,z)\Big]\\
&+z\pi iI_{b-1}(zu)\left[\sum_{s=0}^{N-1}\frac{\tilde{A}_s(e^{i\theta}z)}{t^{2s}}+g_4(t,\theta,ze^{i\pi})\right]\\
&+\frac{z}{t}\pi iI_b(zu)\left[\sum_{s=0}^{N-1}\frac{\tilde{B}_s(e^{i\theta}z)}{t^{2s}}+h_4(t,\theta,ze^{i\pi})\right]\\
&=\beta_2(u)\frac{\pi i2^bu^{1-b}}{\Gamma(1+a-b)}\Bigg[zI_{b-1}(uz)(\sum_{s=0}^{N-1}\frac{\tilde{A}_s(e^{i\theta}z)}{t^{2s}}+g_2(t,e^{i\theta}z))\\
&+\frac{z}{t}I_b(\sum_{s=0}^{N-1}\frac{\tilde{B}_s(e^{i\theta}z)}{t^{2s}}+e^{i\theta}zh_2(t,e^{i\theta}z))\Bigg]
\end{aligned}\tag{A.4.4}$$

Consider the asymptotic expansions of $I_v(uz)=\frac{e^{uz}}{\sqrt{2\pi uz}}(1+\mathcal{O}(\frac{1}{uz}))$ and $K_v(uz)=\sqrt{\frac{\pi}{2uz}}e^{-uz}(1+\mathcal{O}(\frac{1}{uz}))$ (Since $|\arg(uz)|=\theta\le\frac{\pi}{2}-\delta$, the expansions are valid). Divide both sides of Equation (A.4.4) by $zI_\mu(uz)(\sum_{s=0}^{N-1}\frac{\tilde{A}_s(e^{i\theta}z)}{t^{2s}}+g_2(t,e^{i\theta}z))+\frac{z}{t}I_{\mu+1}(\sum_{s=0}^{N-1}\frac{\tilde{B}_s(e^{i\theta}z)}{t^{2s}}+e^{i\theta}zh_2(t,e^{i\theta}z))$, to get the desired property. □

A.5 PROOF OF LEMMA 3.2.7

Proof. When $\Re(b)\ge 1$, apply lemma 1.5 and lemma 1.6 directly. When $\Re(b)<1$, use a method similar to that of Volkmer (Volkmer et al., 2016):

Because $\frac{\tilde{A}(e^{i\theta})}{t^{2s}}=\frac{A_s(z)}{u^{2s}}$, $\frac{\tilde{B}(e^{i\theta})}{t^{2s+1}}=\frac{A_s(z)}{u^{2s+1}}$, when $\Re(b)\ge 1$, we can write the expansion as:

$$\begin{aligned}
&\Gamma(1+\frac{1}{4}u^2-\frac{1}{2}b)2^{-b}u^{b-1}e^{-\frac{1}{2}z^2}z^bU(\frac{1}{4}u^2+\frac{1}{2}b,b,z^2)\\
=&zK_{b-1}(zu)\left[\sum_{s=0}^{N-1}\frac{A_s(z)}{u^{2s}}+g_5(t,\theta,z)\right]-\frac{z}{t}K_b(zu)\left[\sum_{s=0}^{N-1}\frac{B_s(z)}{u^{2s}}+zh_5(t,\theta,z)\right]
\end{aligned}\tag{A.5.1}$$

Let $\mu=b-1$, If we define $a_0(z)=1$, and $a_{s+1}(z)=A_{s+1}(-\mu,z)+\frac{2\mu}{z}B_s(-\mu,z)$, $b_s=B_s(-\mu,z)$. (Notice here A_s and B_s here are polynomials in z with coefficient involving μ.) We list some results of Volkmer (Volkmer et al., 2016):

$$\left[1+2\mu\sum_{s=0}^{\infty}\frac{B_s'(-\mu,0)}{u^{2s+2}}\right]\sum_{s=0}^{\infty}\frac{A_s(z)}{u^{2s}}=\frac{a_s(z)}{u^{2s}}\tag{A.5.2}$$

$$\left[1+2\mu\sum_{s=0}^{\infty}\frac{B_s'(-\mu,0)}{u^{2s+2}}\right]\sum_{s=0}^{\infty}\frac{B_s(z)}{u^{2s}}=\frac{b_s(z)}{u^{2s}} \tag{A.5.3}$$

$$\left[1-2\mu\sum_{s=0}^{\infty}\frac{B_s'(\mu,0)}{u^{2s+2}}\right]\sum_{s=0}^{\infty}\frac{a_s(z)}{u^{2s}}=\frac{A_s(z)}{u^{2s}} \tag{A.5.4}$$

$$\left[1-2\mu\sum_{s=0}^{\infty}\frac{B_s'(-\mu,0)}{u^{2s+2}})\right]\sum_{s=0}^{\infty}\frac{b_s(z)}{u^{2s}}=\frac{B_s(z)}{u^{2s}} \tag{A.5.5}$$

$$\left[1-2\mu\sum_{s=0}^{\infty}\frac{B_s'(\mu,0)}{u^{2s+2}})\right]\left[1+2\mu\sum_{s=0}^{\infty}\frac{B_s'(-\mu,0)}{u^{2s+2}}\right]=1 \tag{A.5.6}$$

From Lemma 3.2.2, Lemma 3.2.6 and Equation (A.5.6), we can prove that for all b and all N,

$$\frac{\Gamma(1+a-b)}{\Gamma(a)}2^{2-2b}u^{2b-2}=1+2(1-b)\sum_{s=0}^{N-1}\frac{B_s'(\mu,0)}{u^{2s+2}}+\mathcal{O}(\frac{1}{t^{2N+2}}) \tag{A.5.7}$$

With Equation (A.5.2) to (A.5.6), we know the expansion (A.5.1) is equivalent to the following expansion:

$$\begin{aligned}&\Gamma(\frac{1}{4}u^2+\frac{1}{2}b)2^{b-2}u^{1-b}e^{-\frac{1}{2}z^2}z^bU(\frac{1}{4}u^2+\frac{1}{2}b,b,z^2)\\&=zK_{b-1}(zu)\left[\sum_{s=0}^{N-1}\frac{a_s(z)}{u^{2s}}+g_5(t,\theta,z)\right]\\&-\frac{z}{t}K_b(zu)\left[\sum_{s=0}^{N-1}\frac{b_s(z)}{u^{2s}}+zh_5(t,\theta,z)\right]\end{aligned} \tag{A.5.8}$$

Then for the case $\Re(b)<1$, by (Olver, 2010):

$$U(a,b,z^2)=z^{2-2b}U(1+a-b,2-b,z^2) \tag{A.5.9}$$

We know

$$\begin{aligned}&\Gamma(\frac{1}{4}u^2+\frac{1}{2}b)2^{b-2}u^{1-b}e^{-\frac{1}{2}z^2}z^bU(\frac{1}{4}u^2+\frac{1}{2}b,b,z^2)\\&=zK_{-\mu}(zu)\left[\sum_{s=0}^{N-1}\frac{A_s(-\mu,z)}{u^{2s}}+g_5(t,\theta,z)\right]\\&-\frac{z}{t}K_{-\mu+1}(zu)\left[\sum_{s=0}^{N-1}\frac{B_s(-\mu,z)}{u^{2s}}+zh_5(t,\theta,z)\right]\end{aligned} \tag{A.5.10}$$

On the other hand, by the definition of $a_s(z)$ and $b_s(z)$, along with $K_v(x)=K_{-v}(x)$, $K_{v-1}(x)-K_{v+1}(x)=-\frac{2v}{x}K_v(x)$, we can rewrite Equation (A.5.10) as Equation (A.5.9), which is equivalent with expansion (A.5.1). Thus, we know expansion (3.2.7) is valid for all $\Re(b)>0$. □

APPENDIX B

PROOF IN SECTION 3.3

B.1 PROOF OF LEMMA 3.3.1

Proof. We need to show that with different parameters, the Laplace transforms are different.

Case 1: $y < y_c$

Consider $\frac{F(\alpha_1 s, \alpha_2, \alpha_3)}{F(\alpha_1 s, \alpha_2, \alpha_4)}$ with $\alpha_1 > 0$, α_2, $0 < \alpha_3 < \alpha_4$. When $s \to +\infty$, notice all the numbers are real, with Theorem 3.2.8 ($\theta = 0$, $\tilde{A}_0(z) = 1$, $\tilde{B}_0(z) = \frac{1}{6}z^3$) we have:

$$\frac{F(\alpha_1 s, \alpha_2, \alpha_3)}{F(\alpha_1 s, \alpha_2, \alpha_4)} = e^{\frac{1}{2}(\alpha_3 - \alpha_4)} (\frac{\alpha_3}{\alpha_4})^{\frac{1-\alpha_2}{2}} \times \frac{I_{\alpha_2 - 1}(\sqrt{(4\alpha_1 s - 2\alpha_2)\alpha_3})(1 + \mathcal{O}(\frac{1}{s})) + \frac{1}{\sqrt{4\alpha_1 s - 2\alpha_2}} I_{\alpha_2}(\sqrt{(4\alpha_1 s - 2\alpha_2)\alpha_3})(\frac{1}{6}\alpha_3^{\frac{3}{2}} + \mathcal{O}(\frac{1}{s}))}{I_{\alpha_2 - 1}(\sqrt{(4\alpha_1 s - 2\alpha_2)\alpha_4})(1 + \mathcal{O}(\frac{1}{s})) + \frac{1}{\sqrt{4\alpha_1 s - 2\alpha_2}} I_{\alpha_2}(\sqrt{(4\alpha_1 s - 2\alpha_2)\alpha_4})(\frac{1}{6}\alpha_4^{\frac{3}{2}} + \mathcal{O}(\frac{1}{s}))} \tag{B.1.1}$$

On the other hand, modified Bessel functions have these expansion in (Olver, 2014):

$$K_v = (\frac{\pi}{2z})^{\frac{1}{2}} e^{-z} \left(\sum_{s=0}^{n-1} \frac{a_s(v)}{z^s} + \gamma_n \right) \tag{B.1.2}$$

$$I_v(z) = \frac{e^z}{(2\pi z)^{\frac{1}{2}}} \left(\sum_{s=0}^{n-1} (-1)^s \frac{a_s(v)}{z^s} + \delta_n \right) - ie^{-v\pi i} \frac{e^{-z}}{(2\pi z)^{\frac{1}{2}}} \left(\sum_{s=0}^{n-1} \frac{a_s(v)}{z^s} + \gamma_n \right) \tag{B.1.3}$$

$|\gamma_n|$ is bounded by

$$2e^{|(v^2 - \frac{1}{4})z^{-1}|} |a_n(v) z^{-n}| \quad \text{if} \ |arg(z)| \le \frac{1}{2}\pi$$

$|\delta_n|$ is bounded by

$$2\chi(n)e^{\frac{1}{2}\pi|(v^2-\frac{1}{4})z^{-1}|}|a_n(v)z^{-n}| \text{ if } -\frac{1}{2}\pi \le |arg(z)| \le 0$$

$$2\chi(n)e^{\frac{1}{2}\pi|(v^2-\frac{1}{4})(\Re(z))^{-1}|}|a_n(v)(Re(z))^{-n}| \text{ if } 0 \le |arg(z)| < \frac{1}{2}\pi$$

where $\chi(n) = \pi^{\frac{1}{2}}\Gamma(\frac{n}{2}+1)/\Gamma(\frac{1}{2}n+\frac{1}{2})$, $a_n(v) = \frac{\Pi_{k=0}^{n}(4v^2-(2k+1)^2)}{(n+1)!} \times (\sum_{k=0}^{n} \frac{1}{4v^2-(2k+1)^2})$. So we can write:

$$\begin{aligned}\frac{F(\alpha_1 s,\alpha_2,\alpha_3)}{F(\alpha_1 s,\alpha_2,\alpha_4)} &= e^{\frac{1}{2}(\alpha_3-\alpha_4)}\left(\frac{\alpha_3}{\alpha_4}\right)^{\frac{1-2\alpha_2}{4}} \frac{e^{\sqrt{(4\alpha_1 s-2\alpha_2)\alpha_3}}(1+\mathcal{O}(s^{-1/2}))}{e^{\sqrt{(4\alpha_1 s-2\alpha_2)\alpha_4}}(1+\mathcal{O}(s^{-1/2}))} \\ &= e^{\frac{1}{2}(\alpha_3-\alpha_4)}\left(\frac{\alpha_3}{\alpha_4}\right)^{\frac{1-2\alpha_2}{4}} e^{\sqrt{4\alpha_1 s-2\alpha_2}(\sqrt{\alpha_3}-\sqrt{\alpha_4})}(1+\mathcal{O}(\frac{1}{\sqrt{s}}))\end{aligned} \tag{B.1.4}$$

On the other hand, from (Linetsky, 2004), the survival function of T_{y_c} can be written as $P(T_{y_c} > t) = \sum_{n=1}^{\infty} c_n e^{-\lambda_n t}$ with $0 < \lambda_1 < \lambda_2 < ... < \lambda_n \to \infty$, where for large n,

$$\begin{aligned}\lambda_n &\sim \frac{\alpha\pi^2}{\frac{8\alpha y_c}{k^2}}\left(n+\frac{\beta}{2k^2}-\frac{3}{4}\right)^2 - \frac{\alpha\beta}{2k^2} \\ &= \frac{\pi^2}{4\alpha_1\alpha_4}\left(n+\frac{\alpha_2}{2}-\frac{3}{4}\right)^2 - \frac{\alpha_2}{2\alpha_1}\end{aligned} \tag{B.1.5}$$

$$\begin{aligned}c_n \sim \;& \frac{(-1)^{n+1}2\pi(n+\frac{\alpha_2}{2}-\frac{3}{4})}{\pi^2(n+\frac{\alpha_2}{2}-\frac{3}{4})^2-2\alpha_2\alpha_4}\times \\ & e^{\frac{1}{2}(\alpha_3-\alpha_4)}(\frac{\alpha_3}{\alpha_4})^{\frac{1}{4}-\frac{\alpha_2}{2}}\cos\left[\pi\left(n+\frac{\alpha_2}{2}-\frac{3}{4}\right)\sqrt{\frac{\alpha_3}{\alpha_4}}-\frac{\pi\alpha_2}{2}+\frac{\pi}{2}\right]\end{aligned} \tag{B.1.6}$$

To proceed, we need another lemma:

Lemma B.1.1. *If c_n and λ_n satisfy the asymptotic properties (B.1.5) and (B.1.6) (we allow different α_1, α_2, α_3, α_4), the expansion is unique.*

Proof. If we have two expansions with coefficients (c_n, λ_n) and (c'_n, λ'_n):

$$\sum_{n=1}^{\infty} c_n e^{-\lambda_n t} = \sum_{n=1}^{\infty} c'_n e^{-\lambda'_n t}$$

Notice we can assume all the c_n, λ_n, c'_n and λ'_n are not zero. If $\lambda_1 \neq \lambda'_1$, assume $\lambda_1 < \lambda'_1$ then:

$$\frac{c_1}{c'_1} e^{(\lambda'_1 - \lambda_1)t} = \frac{1 + \sum_{n=2}^{\infty} \frac{c'_n}{c'_1} e^{-(\lambda'_n - \lambda'_1)t}}{1 + \sum_{n=2}^{\infty} \frac{c_n}{c_1} e^{-(\lambda_n - \lambda_1)t}}$$

Because of equation (B.1.5) and (B.1.6), we know there exists $N > 3$, $c > 0$, $d > 0$ such that:

$$|\sum_{n=N}^{\infty} \frac{c_n}{c_1} e^{-(\lambda_n - \lambda_1)t}| \leq \sum_{n=N}^{\infty} d e^{-c(n-1)^2 t} < d \int_{N-2}^{\infty} e^{-cl^2 t} dl = d \int_{\frac{N-2}{\sqrt{ct}}}^{\infty} \frac{e^{-s^2}}{\sqrt{ct}} ds$$

By which we know as $t \to \infty$, $\sum_{n=N}^{\infty} \frac{c_n}{c_1} e^{-(\lambda_n - \lambda_1)t} \to 0$. Then $t \to \infty$, $\sum_{n=2}^{N} \frac{c_n}{c_1} e^{-(\lambda_n - \lambda_1)t} \to 0$. So $\sum_{n=2}^{\infty} \frac{c_n}{c_1} e^{-(\lambda_n - \lambda_1)t} \to 0$. With the same method, $\sum_{n=2}^{\infty} \frac{c'_n}{c'_1} e^{-(\lambda'_n - \lambda'_1)t} \to 0$, Therefore $\frac{1 + \sum_{n=2}^{\infty} \frac{c'_n}{c'_1} e^{-(\lambda'_n - \lambda'_1)t}}{1 + \sum_{n=2}^{\infty} \frac{c_n}{c_1} e^{-(\lambda_n - \lambda_1)t}} \to 1$. However, $\frac{c_1}{c'_1} e^{(\lambda'_1 - \lambda_1)t} \to \infty$, we have a contradiction. So $\lambda_1 = \lambda'_1$, $c_1 = c'_1$. For $n \geq 2$, use induction method. $\square$

So if we have two sets of parameters $(\alpha_{11}, \alpha_{21}, \alpha_{31}, \alpha_{41})$ and $(\alpha_{12}, \alpha_{22}, \alpha_{32}, \alpha_{42})$ that makes the Laplace transform the same in s, we must have:

$$\begin{cases} \sqrt{\alpha_{11}}(\sqrt{\alpha_{31}} - \sqrt{\alpha_{41}}) = \sqrt{\alpha_{12}}(\sqrt{\alpha_{32}} - \sqrt{\alpha_{42}}) \\ \sqrt{\alpha_{21}}(\sqrt{\alpha_{31}} - \sqrt{\alpha_{41}}) = \sqrt{\alpha_{22}}(\sqrt{\alpha_{32}} - \sqrt{\alpha_{42}}) \\ \alpha_{11}\alpha_{41} = \alpha_{12}\alpha_{42} \end{cases} \tag{B.1.7}$$

From the first two equations, we know $\frac{\alpha_{11}}{\alpha_{21}} = \frac{\alpha_{12}}{\alpha_{22}}$, along with the 3rd Equation of (B.1.7), we can have $\frac{\alpha_{31}}{\alpha_{41}} = \frac{\alpha_{32}}{\alpha_{42}}$. On the other hand, If we take $s = \frac{\alpha_{21}}{\alpha_{11}} = \frac{\alpha_{22}}{\alpha_{12}}$, we will have $\frac{F(\alpha_{11}s, \alpha_{21}, \alpha_{31})}{F(\alpha_{11}s, \alpha_{21}, \alpha_{41})} = e^{\alpha_{31} - \alpha_{41}} = \frac{F(\alpha_{12}s, \alpha_{22}, \alpha_{32})}{F(\alpha_{12}s, \alpha_{22}, \alpha_{42})} = e^{\alpha_{32} - \alpha_{42}}$. So $\alpha_{31} - \alpha_{41} = \alpha_{32} - \alpha_{42}$. Therefore, we must have $\alpha_{31} = \alpha_{32}$, $\alpha_{41} = \alpha_{42}$. Moreover, by the first and second equation in (A.6.7) again we know $\alpha_{11} = \alpha_{12}$, $\alpha_{21} = \alpha_{22}$

Case 2: $y > y_c$

Instead of F, we deal with $\frac{U(\alpha_1 s,\alpha_2,\alpha_3)}{U(\alpha_1 s,\alpha_2,\alpha_4)}$ with $\alpha_1 > 0$, $\alpha_2 > 0$, $0 < \alpha_4 < \alpha_3$. When $s \to +\infty$, with Theorem 1.8 ($\theta = 0$, $\tilde{A}_0(z) = 1$, $\tilde{B}_0(z) = \frac{1}{6}z^3$) we have:

$$\frac{U(\alpha_1 s, \alpha_2, \alpha_3)}{U(\alpha_1 s, \alpha_2, \alpha_4)} = e^{\frac{1}{2}(\alpha_3-\alpha_4)}(\frac{\alpha_3}{\alpha_4})^{\frac{1-\alpha_2}{2}} \times$$
$$\frac{K_{\alpha_2-1}(\sqrt{(4\alpha_1 s - 2\alpha_2)\alpha_3})(1+\mathcal{O}(\frac{1}{s})) - \frac{1}{\sqrt{4\alpha_1 s-2\alpha_2}}K_{\alpha_2}(\sqrt{(4\alpha_1 s - 2\alpha_2)\alpha_3})(\frac{1}{6}\alpha_3^{\frac{3}{2}} + \mathcal{O}(\frac{1}{s}))}{K_{\alpha_2-1}(\sqrt{(4\alpha_1 s - 2\alpha_2)\alpha_4})(1+\mathcal{O}(\frac{1}{s})) - \frac{1}{\sqrt{4\alpha_1 s-2\alpha_2}}K_{\alpha_2}(\sqrt{(4\alpha_1 s - 2\alpha_2)\alpha_4})(\frac{1}{6}\alpha_4^{\frac{3}{2}} + \mathcal{O}(\frac{1}{s}))} \tag{B.1.8}$$

With the expansion of K, we have:

$$\begin{aligned}\frac{U(\alpha_1 s, \alpha_2, \alpha_3)}{U(\alpha_1 s, \alpha_2, \alpha_4)} &= e^{\frac{1}{2}(\alpha_3-\alpha_4)}(\frac{\alpha_3}{\alpha_4})^{\frac{1-2\alpha_2}{4}} \frac{e^{-\sqrt{(4\alpha_1 s-2\alpha_2)\alpha_3}}(1+\mathcal{O}(\frac{1}{\sqrt{s}}))}{e^{-\sqrt{(4\alpha_1 s-2\alpha_2)\alpha_4}}(1+\mathcal{O}(\frac{1}{\sqrt{s}}))} \\ &= e^{\frac{1}{2}(\alpha_3-\alpha_4)}\left(\frac{\alpha_3}{\alpha_4}\right)^{\frac{1-2\alpha_2}{4}} e^{\sqrt{4\alpha_1 s-2\alpha_2}(\sqrt{\alpha_4}-\sqrt{\alpha_3})}\left(1+\mathcal{O}(\frac{1}{\sqrt{s}})\right)\end{aligned} \tag{B.1.9}$$

Meanwhile, using the result from (Linetsky, 2004) again, we know the survival function of T_{y_c} can be written as $P(T_{y_c} > t) = \sum_{n=1}^{\infty} c_n e^{-\lambda_n t}$ with $0 < \lambda_1 < \lambda_2 < ... < \lambda_n \to \infty$, where for large n,

$$\lambda_n \sim \frac{1}{\alpha_1}(k_n - \frac{\alpha_2}{2}) \tag{B.1.10}$$

where

$$k_n \sim n - \frac{1}{4} + \frac{2\alpha_4}{\pi^2} + \frac{2}{\pi}\sqrt{(n-\frac{1}{4})\alpha_4 + \frac{\alpha_4^2}{\pi^2}} \tag{B.1.11}$$

and

$$c_n \sim \frac{(-1)^{n+1}\sqrt{k_n}}{(k_n - \frac{b}{2})(\pi\sqrt{k_n} - \sqrt{\alpha_4})} e^{\frac{1}{2}(\alpha_3-\alpha_4)}(\frac{\alpha_3}{\alpha_4})^{\frac{1}{4}-\frac{\alpha_2}{2}} \cos(2\sqrt{k_n\alpha_3} - \pi k_n + \frac{\pi}{4}) \tag{B.1.12}$$

By a similar procedure of proving Lemma (B.1.1), we know the expansion for the same survival functions is unique.

Again, if we have two sets of parameters $(\alpha_{11}, \alpha_{21}, \alpha_{31}, \alpha_{41})$ and $(\alpha_{12}, \alpha_{22}, \alpha_{32}, \alpha_{42})$ that makes the Laplace transform the same in s, we should have:

$$\begin{cases} \sqrt{\alpha_{11}}(\sqrt{\alpha_{31}} - \sqrt{\alpha_{41}}) = \sqrt{\alpha_{12}}(\sqrt{\alpha_{32}} - \sqrt{\alpha_{42}}) \\ \sqrt{\alpha_{21}}(\sqrt{\alpha_{31}} - \sqrt{\alpha_{41}}) = \sqrt{\alpha_{22}}(\sqrt{\alpha_{32}} - \sqrt{\alpha_{42}}) \\ \alpha_{11} = \alpha_{12} \end{cases} \tag{B.1.13}$$

Thus, obviously we will have $\alpha_{11} = \alpha_{12}$, $\alpha_{21} = \alpha_{22}$, $\alpha_{31} = \alpha_{32}$, $\alpha_{41} = \alpha_{42}$.

Case 3: we still need to check whether the situation in case 1 and case 2 can result in the same Laplace transform. Thus, whether there are two sets of parameters $(\alpha_{11}, \alpha_{21}, \alpha_{31}, \alpha_{41})$ and $(\alpha_{12}, \alpha_{22}, \alpha_{32}, \alpha_{42})$ that make $\frac{F(\alpha_{11}s, \alpha_{21}, \alpha_{31})}{F(\alpha_{11}s, \alpha_{21}, \alpha_{41})} = \frac{U(\alpha_{12}s, \alpha_{22}, \alpha_{32})}{U(\alpha_{12}s, \alpha_{22}, \alpha_{42})}$. By the asymptotic form of λ_n in both cases (Equation (B.1.5) and (B.1.10)), we know the survival functions will be different, therefore, the Laplace transform must be different. $\square$

APPENDIX C

PROOF IN SECTION 3.4

C.1 PROOF OF LEMMA 3.4.1

Proof. Notice that $\alpha_{01}s$ always appears as an integrity, so we only need to consider a small neighborhood of α_{02} and α_{03}. In other words, we consider the behavior of $1 + y(u, \alpha_2, \alpha_3) = 1 + x(s, \alpha_1, \alpha_2, \alpha_3)$ with $|u| \to \infty$. In a small neighborhood of α_{02} in the complex plane, we can find c s.t. $\Re(c) > 0$ and $\Re(c)$ being less than the real part of the whole neighborhood of α_{02}. So by Equation (13.4.2) in (Olver, 2010), we know that

$$1 + y(u, \alpha_2, \alpha_3) = \frac{\frac{\Gamma(\alpha_2)}{\Gamma(\alpha_2-c)} \int_0^1 \frac{F(\alpha_1 s, c, \alpha_3 l)}{\Gamma(c)} l^{c-1}(1-l)^{\alpha_2-c-1} dl}{\frac{\Gamma(\alpha_2)}{\pi^{\frac{1}{2}} 2^{\frac{3}{2}-\alpha_2} u^{\alpha_2-\frac{1}{2}}} e^{\frac{1}{2}\alpha_3} \alpha_3^{\frac{1-2\alpha_2}{4}} e^{u\sqrt{\alpha_3}}} \tag{C.1.1}$$

Below, whenever we must take a square root, we always choose the root with real part greater than 0. Using the expansion in Theorem (3.2.8) again, we know Equation (C.1.1) is equal to:

$$\frac{\frac{\Gamma(\alpha_2)}{\Gamma(\alpha_2-c)} \int_0^1 \frac{\Gamma(c)}{\Gamma(c)\pi^{\frac{1}{2}} 2^{\frac{3}{2}} v^{c-\frac{3}{2}}} e^{\frac{1}{2}\alpha_3 l} (\alpha_3 l)^{\frac{1-2c}{4}} e^{v\sqrt{\alpha_3 l}} (1 + y(v, c, \alpha_3 l)) l^{c-1} (1-l)^{\alpha_2-c-1}}{\frac{\Gamma(\alpha_2)}{\pi^{\frac{1}{2}} 2^{\frac{3}{2}-\alpha_2} u^{\alpha_2-\frac{1}{2}}} e^{\frac{1}{2}\alpha_3} \alpha_3^{\frac{1-2\alpha_2}{4}} e^{u\sqrt{\alpha_3}}} \tag{C.1.2}$$

with $\frac{1}{4}u^2 + \frac{1}{2}\alpha_2 = \frac{1}{4}v^2 + \frac{1}{2}c = \alpha_1 s$. Since c is universal for the neighborhood, by Theorem 3.2.8, $|1 + y(v, c, \alpha_3 l)| \leq K$ for $|v| \geq$ some t_1. Take the absolute value of both sides of Equation (C.1.2):

$$\begin{aligned} |1 + y(u, \alpha_2, \alpha_3)| \leq & \frac{K}{|\Gamma(\alpha_2 - c)|} |\alpha_3^{\frac{\alpha_2-c}{2}}| |2^{c-\alpha_2}| \frac{|u^{\alpha_2-\frac{1}{2}}|}{|v^{c-\frac{1}{2}}|} |e^{-u\sqrt{\alpha_3}}| \\ & \left| \int_0^1 e^{v\sqrt{\alpha_3 l}} l^{\frac{c}{2}-\frac{3}{4}} (1-l)^{\alpha_2-c-1} e^{\frac{\alpha_3 l - \alpha_3}{2}} dl \right|. \end{aligned} \tag{C.1.3}$$

Along with Equation (13.4.1) in (Olver, 2010), consider:

$$
\begin{aligned}
&|\int_0^1 e^{v\sqrt{\alpha_3 l}} l^{\frac{c}{2}-\frac{3}{4}}(1-l)^{\alpha_2-c-1}e^{\frac{\alpha_3 l-\alpha_3}{2}}dl| \\
\leq &\int_0^1 e^{\Re(v\sqrt{\alpha_3})\sqrt{l}} l^{\Re(\frac{c}{2}-\frac{3}{4})}(1-l)^{\Re(\alpha_2-c-1)}dl \\
= &\int_0^1 e^{\Re(v\sqrt{\alpha_3})l} l^{\Re(c-\frac{3}{2})}(1-l)^{\Re(\alpha_2-c-1)}(1+l)^{\Re(\alpha_2-c-1)}(2l)dl \\
\leq &2^{\Re(\alpha_2-c)}\int_0^1 e^{\Re(v\sqrt{\alpha_3})l} l^{\Re(c-\frac{1}{2})}(1-l)^{\Re(\alpha_2-c-1)}dl \\
= &2^{\Re(\alpha_2-c)}\frac{\Gamma(\Re(c)+\frac{1}{2})\Gamma(\Re(\alpha_2-c))}{\Gamma(\Re(\alpha_2)+\frac{1}{2})}F(\Re(c)+\frac{1}{2},\Re(\alpha_2)+\frac{1}{2},\Re(v\sqrt{\alpha_3}))
\end{aligned}
\tag{C.1.4}
$$

So now we turn to $F(\Re(c)+\frac{1}{2},\Re(\alpha_2)+\frac{1}{2},\Re(v\sqrt{\alpha_3})$ as $v\to\infty$. when $\Re(z)>|b-2a|$, by Equation (13.2.41), (13.7.4), (13.7.5), (13.7.8) and (13.7.9) in (Olver, 2010):

$$
\frac{1}{\Gamma(b)}F(a,b,z)=\frac{e^{-a\pi i}}{\Gamma(b-a)}U(a,b,z)+\frac{e^{(b-a)\pi i}}{\Gamma(a)}e^zU(b-a,b,e^{\pi i}z)
$$

$$
U(a,b,z)=z^{-a}+\epsilon_1(z)
$$

Where $|\epsilon_1(z)|\leq 2\alpha|\frac{a(a-b+1)}{z^{a+1}}|e^{\frac{2\alpha\rho}{|z|}}$, $\alpha=\frac{1}{1-|\frac{b-2a}{z}|}$, $\rho=\frac{1}{2}|2a^2-2ab+b|+\frac{\sigma(1+\frac{1}{4}\sigma)}{|1-\sigma|^2}$, $\sigma=|\frac{b-2a}{2z}|$. Thus:

$$
\begin{aligned}
|F(a,b,z)|\leq &|\Gamma(b)|\Bigg(\frac{1}{|\Gamma(b-a)|}(|z^{-a}|+|\epsilon_1(z)|)+\frac{1}{|\Gamma(a)|}|e^z|(|z^{a-b}|+|\hat{\epsilon(z)}|)\Bigg) \\
\leq &|\Gamma(b)|\Bigg(\frac{1}{|\Gamma(b-a)|}\left[|z^{-a}|+\frac{2}{1-|\frac{b-2a}{z}|}|\frac{a(a-b+1)}{z^{a+1}}|e^{\frac{\frac{2}{1-|\frac{b-2a}{z}|}(\frac{1}{2}|2a^2-2ab+b|+\frac{\sigma(1+\frac{1}{4}\sigma)}{|1-\sigma|^2})}{|z|}}\right] \\
&+\frac{1}{|\Gamma(a)|}|e^z|\left[|z^{a-b}|+\frac{2}{1-|\frac{2a-b}{z}|}|\frac{(b-a)(1-a)}{z^{b-a+1}}|e^{\frac{\frac{2}{1-|\frac{2a-b}{z}|}(\frac{1}{2}|2(b-a)^2-2(b-a)b+b|+\frac{\sigma(1+\frac{1}{4}\sigma)}{|1-\sigma|^2})}{|z|}}\right]\Bigg)
\end{aligned}
\tag{C.1.5}
$$

Then it is not hard to observe that when $|v|$ large enough, in a neighborhood of α_{02}, α_{03}, there exists $0<L<\infty$, s.t.

$$
|F(\Re(c)+\frac{1}{2},\Re(\alpha_2)+\frac{1}{2},\Re(v\sqrt{\alpha_3})|\leq Le^{\Re(v\sqrt{\alpha_3})}(\Re(v\sqrt{\alpha_3}))^{\Re(c-\alpha_2)}
\tag{C.1.6}
$$

Together with Equation (C.1.1), (C.1.2) and (C.1.6), we are able to obtain:

$$\begin{aligned}|1+y(u,\alpha_2,\alpha_3)| \le \frac{KL}{|\Gamma(\alpha_2-c)|}|\alpha_3^{\frac{\alpha_2-c}{2}}||2^{c-\alpha_2}|\frac{|u^{\alpha_2-\frac{1}{2}}|}{|v^{c-\frac{1}{2}}|}|e^{-u\sqrt{\alpha_3}}|2^{\Re(\alpha_2-c)}\times \\ \frac{\Gamma(\Re(c)+\frac{1}{2})\Gamma(\Re(\alpha_2-c))}{\Gamma(\Re(\alpha_2)+\frac{1}{2})}e^{\Re(v\sqrt{\alpha_3})}(\Re(v\sqrt{\alpha_3}))^{\Re(c-\alpha_2)}\end{aligned} \tag{C.1.7}$$

Notice again $u=te^{i\theta}$, $v=t'e^{i\theta'}$, $\sqrt{\alpha_3}=t''e^{i\theta''}$ and $\frac{1}{4}u^2+\frac{1}{2}\alpha_2=\frac{1}{4}v^2+\frac{1}{2}c=\alpha_1 s$, then

$$|u^{\alpha_2-\frac{1}{2}}| = t^{\Re(\alpha_2-\frac{1}{2})}e^{-\Im(\alpha_2)\theta}$$

$$|v^{c-\frac{1}{2}}| = (t')^{\Re(c-\frac{1}{2})}e^{-\Im(c)\theta'}$$

$$(\Re(v\sqrt{\alpha_3}))^{\Re(c-\alpha_2)} = (t't''\cos(\theta'+\theta''))^{\Re(c-\alpha_2)}$$

By which we can have:

$$|\frac{|u^{\alpha_2-\frac{1}{2}}|}{|v^{c-\frac{1}{2}}|}|(\Re(v\sqrt{\alpha_3}))^{\Re(c-\alpha_2)} = (\frac{t}{t'})^{\Re(\alpha_2-\frac{1}{2})}e^{\Im(c)\theta'-\Im(\alpha_2\theta)}\cos(\theta'+\theta'')^{\Re(c)-\Re(\alpha_2)}(t'')^{\Re(c-\alpha_2)}$$

As $|u|\to\infty$, $\frac{t}{t'}$ will increase to 1, θ' is in $(-\frac{\pi}{4},\frac{\pi}{4})$, $|\theta''|$ can be really small in the whole neighbourhood of α_{03}. It is obvious in a neighborhood of α_{02} and α_{03}, there exists $L'<\infty$ s.t.

$$|\frac{|u^{\alpha_2-\frac{1}{2}}|}{|v^{c-\frac{1}{2}}|}|(\Re(v\sqrt{\alpha_3}))^{\Re(c-\alpha_2)} \le L'$$

On the other hand, $e^{\Re(v\sqrt{\alpha_3})}|e^{-u\sqrt{\alpha_3}}| = e^{\Re(\sqrt{\alpha_3}(v-u))}$, $v-u=\frac{2\alpha_2-2c}{u+v}\to 0$, so $e^{\Re(v\sqrt{\alpha_3})}|e^{-u\sqrt{\alpha_3}}|\le K'$ in a neighborhood of α_{02} and α_{03}.

With these observations, we have that for $|u|$ (or $|v|$) large enough, $|1+y(u,\alpha_{21},\alpha_{31})|$ is bounded from above in a small neighborhood of α_{02} and α_{03} in the complex plane. For small $|u|$, using the continuity of $|1+y(u,\alpha_2,\alpha_3)|$ and compactness, we know it is still bounded from above. Therefore, in a small neighborhood of α_{02} and α_{03}, there exists a $k_2<\infty$ such that $|1+y(u,\alpha_2,\alpha_3)|\le k_2$.

So far we have shown that $|1+y(u,\alpha_2,\alpha_3)|$ is bounded from above in a neighborhood of α_{02} and α_{03}. Next, we show that it is also bounded from below by a constant greater than 0. We use the technique of proof by contradiction. Suppose that the radius of the neighborhoods of the upper bound case are δ_1 and δ_2 for α_{02} and α_{03}, respectively. Then we shrink the neighborhood to be have radii $\frac{\delta_1}{3}$ and $\frac{\delta_2}{3}$. If not bounded from below by a constant greater than 0, then there exits u_n, $\alpha_{2,n}$, $\alpha_{3,n}$ such that as $|u_n| \to \infty$ and $1 + y(u_n, \alpha_{2,n}, \alpha_{3,n}) \to 0$.

By compactness of the neighborhood, there exits a subsequence of n, n_k such that $\alpha_{2,n_k}, \alpha_{3,n_k} \to \hat{\alpha_2}, \hat{\alpha_3}$, with $\hat{\alpha_2}$ and $\hat{\alpha_3}$ are in the neighborhood of α_{02} and α_{03}. Using the triangle inequality and Rolle's mean value theorem in the complex plane, we have

$$\begin{aligned}
&|1 + y(u_{n_k}, \hat{\alpha_2}, \hat{\alpha_3}) - 1 - y(u_{n_k}, \alpha_{2,n_k}, \alpha_{3,n_k})| \\
\leq &|1 + y(u_{n_k}, \hat{\alpha_2}, \hat{\alpha_3}) - 1 - y(u_{n_k}, \hat{\alpha_2}, \alpha_{3,n_k})| \\
+ &|1 + y(u_{n_k}, \hat{\alpha_2}, \alpha_{3,n_k}) - 1 - y(u_{n_k}, \alpha_{2,n_k}, \alpha_{3,n_k})| \\
\leq &|y(u_{n_k}, \hat{\alpha_2}, \hat{\alpha_3}) - y(u_{n_k}, \hat{\alpha_2}, \alpha_{3,n_k})| \\
+ &|(\hat{\alpha_2} - \alpha_{2,n_k})| \left| \Re(\frac{\partial(1 + y(u_{n_k}, \alpha_2, \alpha_{3,n_k}))}{\partial \alpha_2}|_{\alpha_2 = z_1}) + i\Im(\frac{\partial(1 + y(u_{n_k}, \alpha_2, \alpha_{3,n_k}))}{\partial \alpha_2}|_{\alpha_2 = z_2}) \right|
\end{aligned} \tag{C.1.8}$$

where z_1, z_2 are on the line between α_{2,n_k} and $\hat{\alpha_2}$.

Meanwhile, because of the shrinkage of the neighborhood, we know for any z on the line of $\hat{\alpha_2}$ and α_{2,n_k} in the complex plane, z is in the circle with center α_{02} and radius $\frac{2\delta_1}{3}$, so $|1+y(u_{n_k}, \alpha_2, \alpha_{3,n_k})|$ can be bounded by k_2. With Cauchy's inequality:

$$\left| \frac{\partial(1 + y(u_{n_k}, \alpha_2, \alpha_{3,n_k}))}{\partial \alpha_2}|_{\alpha_2 = z_1} \right| \leq \frac{3k_2}{2\delta_1} \tag{C.1.9}$$

$$\left| \frac{\partial(1 + y(u_{n_k}, \alpha_2, \alpha_{3,n_k}))}{\partial \alpha_2}|_{\alpha_2 = z_2} \right| \leq \frac{3k_2}{2\delta_1} \tag{C.1.10}$$

Based on Theorem (3.2.8) and the analysis in Lemma (3.2.1), for $\hat{\alpha_2}$, when $|u_{n_k}|$ is large enough, there exists $K_1 > 0$ s.t.:

$$|y(u_{n_k}, \hat{\alpha_2}, \hat{\alpha_3})| \leq \left| \frac{K_1}{u_{n_k}} \right|$$

$$|y(u_{n_k}, \hat{\alpha_2}, \hat{\alpha_{3,n_k}})| \leq \left| \frac{K_1}{u_{n_k}} \right|$$

With the observations above, we know the right hand side of inequality (C.1.8) is tending to 0, however, because $y(u_{n_k}, \hat{\alpha_2}, \hat{\alpha_3}) \to 0$, the left hand side of inequality (C.1.8) is not tending to 0, to get a contradiction. □

C.2 PROOF OF LEMMA 3.4.2

Proof. We consider $\hat{g}$ first, it is easy to see: $\hat{g}_1(s|\alpha) = s\hat{g}\times(\frac{F_1(\alpha_1 s,\alpha_2,\alpha_3)}{F(\alpha_1 s,\alpha_2,\alpha_3)} - \frac{F_1(\alpha_1 s,\alpha_2,\alpha_4)}{F(\alpha_1 s,\alpha_2,\alpha_4)})$, $\hat{g}_2(s|\alpha) = \hat{g}\times(\frac{F_2(\alpha_1 s,\alpha_2,\alpha_3)}{F(\alpha_1 s,\alpha_2,\alpha_3)} - \frac{F_2(\alpha_1 s,\alpha_2,\alpha_4)}{F(\alpha_1 s,\alpha_2,\alpha_4)})$, ..., $\hat{g_{11}}(s|\alpha) = s^2\hat{g}\times((\frac{F_1(\alpha_1 s,\alpha_2,\alpha_3)}{F(\alpha_1 s,\alpha_2,\alpha_3)} - \frac{F_1(\alpha_1 s,\alpha_2,\alpha_4)}{F(\alpha_1 s,\alpha_2,\alpha_4)})^2 + \frac{F_{11}(\alpha_1 s,\alpha_2,\alpha_3)}{F(\alpha_1 s,\alpha_2,\alpha_3)} - (\frac{F_1(\alpha_1 s,\alpha_2,\alpha_3)}{F(\alpha_1 s,\alpha_2,\alpha_3)})^2 - \frac{F_{11}(\alpha_1 s,\alpha_2,\alpha_4)}{F(\alpha_1 s,\alpha_2,\alpha_4)} + (\frac{F_1(\alpha_1 s,\alpha_2,\alpha_4)}{F(\alpha_1 s,\alpha_2,\alpha_4)})^2)$, ...One thing can be noticed is that it suffices to show those ratios $\frac{F_i}{F}$, $\frac{F_{ij}}{F}$, and $\frac{F_{ijk}}{F}$ increases at most as a polynomial in $|s|$.

For $\frac{F_1(\alpha_1 s,\alpha_2,\alpha_3)}{F(\alpha_1 s,\alpha_2,\alpha_3)}$, we know that

$$F(\alpha_1 s, \alpha_2, \alpha_3) = \frac{\Gamma(\alpha_2)}{\pi^{\frac{1}{2}} 2^{\frac{3}{2}-\alpha_2} u^{\alpha_2 - \frac{1}{2}}} e^{\frac{1}{2}\alpha_3} \alpha_3^{\frac{1-2\alpha_2}{4}} e^{u\sqrt{\alpha_3}} (1 + y(u, \alpha_2, \alpha_3)) \tag{C.2.1}$$

where $s = \frac{u^2}{4\alpha_1} + \frac{\alpha_2}{2\alpha_1}$, $u = e^{i\theta}$, $y(u, \alpha_2, \alpha_3) = \mathcal{O}(\frac{1}{t})$. So

$$\frac{\frac{\partial F(\alpha_1 s,\alpha_2,\alpha_3)}{\partial u}}{F(\alpha_1 s, \alpha_2, \alpha_3)} = (\frac{1}{2} - \alpha_2)(\frac{1}{u}) + \sqrt{\alpha_3} + \frac{\frac{\partial y(u,\alpha_2,\alpha_3)}{\partial u}}{1 + y(u, \alpha_2, \alpha_3)}$$

Because $y(u, \alpha_2, \alpha_3) = \mathcal{O}(1/t)$, by the Cauchy's inequality (notice here we need asymptotic expansion of $F(\alpha_1 s, \alpha_2, \alpha_3)$ for $|\theta| \geq \frac{\pi}{4} + \delta$, δ is small) we know $\frac{\partial y(u,\alpha_2,\alpha_3)}{\partial u}$ is bounded for $\Re(s) \geq \frac{\alpha_2}{2\alpha_1}$. While $\frac{F_1(\alpha_1 s,\alpha_2,\alpha_3)}{F(\alpha_1 s,\alpha_2,\alpha_3)} = \frac{\frac{\partial F(\alpha_1 s,\alpha_2,\alpha_3)}{\partial u}}{F(\alpha_1 s,\alpha_2,\alpha_3)} \times \frac{\partial u}{\partial(s\alpha_1)}$, we can conclude that $\frac{F_1(\alpha_1 s,\alpha_2,\alpha_3)}{F(\alpha_1 s,\alpha_2,\alpha_3)}$ increases at most polynomial in $|s|$. For $\frac{F_2(\alpha_1 s,\alpha_2,\alpha_3)}{F(\alpha_1 s,\alpha_2,\alpha_3)}$, with the same idea of $\frac{F_1(\alpha_1 s,\alpha_2,\alpha_3)}{F(\alpha_1 s,\alpha_2,\alpha_3)}$, write down $F_2(\alpha_1 s, \alpha_2, \alpha_3)$ based on Equation (C.2.1). Because we have shown the boundedness of $y(u, \alpha_2, \alpha_3)$ in a samll neighborhood of α_2 in the complex plane, by the Cauchy's inequality again, we know $\frac{\partial y(u,\alpha_2,\alpha_3)}{\partial \alpha_2}$ is bounded as $|u| \to \infty$. So we are able to conclude that $\frac{F_2(\alpha_1 s,\alpha_2,\alpha_3)}{F(\alpha_1 s,\alpha_2,\alpha_3)}$ increases at most polynomial in $|s|$.

For $\frac{F_3(\alpha_1 s,\alpha_2,\alpha_3)}{F(\alpha_1 s,\alpha_2,\alpha_3)}$, because of Equation (13.3.15) in (Olver, 2010) we have

$$F_3(\alpha_1 s, \alpha_2, \alpha_3) = \frac{\alpha_1 s}{\alpha_2} F(\alpha_1 s + 1, \alpha_2 + 1, \alpha_3); \tag{C.2.2}$$

by Theorem (3.2.8) and comparing the expansion of $F(\alpha_1 s + 1, \alpha_2 + 1, \alpha_3)$ and $F(\alpha_1 s, \alpha_2, \alpha_3)$, we know $\frac{F_3(\alpha_1 s,\alpha_2,\alpha_3)}{F(\alpha_1 s,\alpha_2,\alpha_3)}$ increases at most polynomial in $|s|$. For higher orders, we repeat the above procedure to get a similar conclusion. □

C.3 PROOF OF LEMMA 3.4.3

Proof. The proof for all cases are similar: basically it is an application of Lemmas (3.4.1) and (3.4.2). We show two of them here: $\hat{g}$ and $\hat{g}_2$. First, we can find t_1 such that $t_1 \geq \sup \frac{\alpha_2'}{2\alpha_1'}$, the supremum is taken over the small neighborhood of α_1 and α_2. Then we know when $|s| \geq t_1$, the expansion in Theorem (3.2.8) is valid:

$$\hat{g} = \left(\frac{\alpha_3}{\alpha_4}\right)^{\frac{1-2\alpha_2}{4}} e^{u(\sqrt{\alpha_3}-\sqrt{\alpha_4})} \frac{1+y(u,\alpha_2,\alpha_3)}{1+y(u,\alpha_2,\alpha_4)}$$

then with Lemma (3.4.1), we know the conclusion is valid for $\hat{g}$. When it comes to $\hat{g}_2$:

$$\hat{g}_2 = s\hat{g} \times \left[\frac{F_2(\alpha_1 s,\alpha_2,\alpha_3)}{F(\alpha_1 s,\alpha_2,\alpha_3)} - \frac{F_2(\alpha_1 s,\alpha_2,\alpha_4)}{F(\alpha_1 s,\alpha_2,\alpha_4)}\right]$$

As we did in the proof of Lemma (3.4.2), write down $F_2(\alpha_1 s,\alpha_2,\alpha_3)$ based on Equation (C.2.1). It is not hard to see the boundedness of $\frac{\partial y(u,\alpha_{21},\alpha_{31})}{\partial \alpha_{21}}$ in a smaller neighborhood by the Cauchy's inequality. Then we are able to show the validity for $\hat{g}_2$. All the other cases can be proved by Lemma (3.4.1), Lemma (3.4.2) and Equation (13.3.15) in (Olver, 2010). □

APPENDIX D

PROOF IN SECTION 3.5

D.1 PROOF OF LEMMA 3.5.1

Proof. We can prove it by induction easily using the recurrence relation:

$$\begin{aligned}
B_{3n} &= \frac{3n+b-2}{3n} B_{3n-2} - \frac{2k}{3n} B_{3n-3} \\
B_{3n+1} &= \frac{3n+b-1}{3n+1} B_{3n-1} - \frac{2k}{3n+1} B_{3n-2} \\
B_{3n+2} &= \frac{3n+b}{3n+2} B_{3n} - \frac{2k}{3n+2} B_{3n-1}
\end{aligned}$$

□

D.2 PROOF OF LEMMA 3.5.2

Proof. According to page 68 of Slater (Slater, 1960), $|J_{b+n-1}(2k^{\frac{1}{2}}x^{\frac{1}{2}})| \leq 1$ when b real and positive and $n = 1, 2, \cdots$. We can see that in a closure of (b, x), there exists r_2 when k large such that For $n \geq 1$:

$$|U_n| \leq r_2 |k|^{-\frac{1}{2}n} \left| \frac{x}{4} \right|^{\frac{3n}{2}}$$

For $n = 0$:

$$|U_n| \leq |J_{b-1}(2k^{\frac{1}{2}}x^{\frac{1}{2}})| + \left| \frac{bx}{8k} \right|$$

Combine the two observations above, we have that:

$$F(a,b,x) = \Gamma(b)e^{x/2}(kx)^{\frac{1}{2}-\frac{1}{2}b}[J_{b-1}(2k^{\frac{1}{2}}x^{\frac{1}{2}}) + R^1(a,b,x)] \tag{D.2.1}$$

with $|R^1(a,b,x)| \leq k_1 k^{-\frac{1}{2}}$. On the other hand, $J_v(\xi) = \sqrt{\frac{2}{\pi\xi}}\cos(\xi - \frac{1}{2}\pi v - \frac{1}{4}\pi)(1 + \mathcal{O}(|\xi^{-1}|))$, where $\mathcal{O}(|\xi^{-1}|)$ can be uniform in a neighbourhood of (v,ξ). □

D.3 PROOF OF LEMMA 3.5.3

Proof. Consider $\frac{\partial u_n}{\partial a}$ $(n \geq 1)$ first:

$$\begin{aligned}
\frac{\partial u_n}{\partial a} &= \frac{\partial B_n(k,\frac{1}{b})}{\partial a}(\frac{x}{4k})^{\frac{1}{2}n} J_{b+n-1}(2k^{\frac{1}{2}}x^{\frac{1}{2}}) \\
&+ B_n(k,\frac{b}{2})(\frac{1}{2}n)(\frac{x}{4k})^{\frac{1}{2}n-1}(-\frac{x}{4})(\frac{1}{k^2})J_{b+n-1}(2k^{\frac{1}{2}}x^{\frac{1}{2}})(-1) \\
&+ B_n(k,\frac{b}{2})(\frac{x}{4k})^{\frac{1}{2}n}(x^{\frac{1}{2}}k^{-\frac{1}{2}})(-1)\frac{\partial J_{b+n-1}(2k^{\frac{1}{2}}x^{\frac{1}{2}})}{\partial 2k^{\frac{1}{2}}x^{\frac{1}{2}}}(2k^{\frac{1}{2}}x^{\frac{1}{2}})
\end{aligned} \tag{D.3.1}$$

Because $\frac{\partial J_{b+n-1}(z)}{\partial z} = \frac{b+n-1}{z}J_{b+n-1}(z) - J_{b+n}(z)$ and B_n are some polynomials in k, we know that for $n \geq 1$ (note that when $-a$ is large, $\frac{\partial u_{3n}}{\partial a}$ is the dominant term):

$$\begin{aligned}
|\frac{\partial U_n}{\partial a}| &= |\frac{\partial u_{3n}}{\partial a} + \frac{\partial u_{3n+1}}{\partial a} + \frac{\partial u_{3n+2}}{\partial a}| \\
&\leq r_3 n^2 |k^{n-1}(\frac{x}{4k})^{\frac{3n}{2}}| + r_4 n |k^n (\frac{x}{4k})^{\frac{3n}{2}-1}\frac{1}{k^2}| \\
&+ r_5 |k^n (\frac{x}{4k})^{\frac{3n}{2}} k^{-\frac{1}{2}}| + r_6 |n k^n (\frac{x}{4k})^{\frac{3n}{2}} k^{-1}| \\
&\leq r_7 n^2 (\frac{x}{4})^{\frac{3n}{2}} |k|^{-\frac{1}{2}n-1} + r_8 (\frac{x}{4})^{\frac{3n}{2}} |k|^{-\frac{1}{2}n-\frac{1}{2}}
\end{aligned} \tag{D.3.2}$$

For $n = 0$,

$$\frac{\partial U_0}{\partial a} = \frac{\partial J_{b-1}(2k^{\frac{1}{2}}x^{\frac{1}{2}})}{\partial a} + \frac{\partial u_2}{\partial a} \tag{D.3.3}$$

So when $-a$ is large (k large), we have uniform convergence for $\sum_{n=0}^{\infty}\frac{\partial U_n}{\partial a}$ with respect to a, we can interchange the integral and differentiation sign. From Equation (D.3.2) and Equation (D.3.3), we know that:

$$\begin{aligned}
\frac{\partial F(a,b,x)}{\partial a} =& \Gamma(b)e^{x/2}(kx)^{\frac{1-b}{2}-1}(-x)\left(\frac{1-b}{2}\right)\left[(\pi x^{\frac{1}{2}}k^{\frac{1}{2}})^{-\frac{1}{2}}\cos(w) + R^2(a,b,x)\right] \\
&+ \Gamma(b)e^{x/2}(kx)^{\frac{1-b}{2}}\left[\frac{\partial J_{b-1}(2k^{\frac{1}{2}}x^{\frac{1}{2}})}{\partial a} + \frac{\partial u_2}{\partial a} + \sum_{n=1}^{\infty}\frac{\partial U_n}{\partial a}\right]
\end{aligned}$$

From Equation (D.3.2) again,

$$|\sum_{n=1}^{\infty}\frac{\partial U_n}{\partial a}| \leq r_7|k|^{-1}\sum_{n=1}^{\infty}n(\frac{x}{4})^{\frac{3n}{2}}|k|^{-\frac{1}{2}n}+r_8k^{-\frac{1}{2}}\frac{\frac{(\frac{x}{4})^{\frac{3}{2}}}{k^{\frac{1}{2}}}}{1-\frac{(\frac{x}{4})^{\frac{3}{2}}}{k^{\frac{1}{2}}}}$$

Along with Equation (D.3.1) for $n = 2$, it is not hard to see that the remainder term can be bounded by term of order k^{-1}. So we just need to investigate:

$$\frac{\partial J_{b-1}(2k^{\frac{1}{2}}x^{\frac{1}{2}})}{\partial a} = (-1)(k^{-\frac{1}{2}}x^{\frac{1}{2}})\left[\frac{b-1}{2k^{\frac{1}{2}}x^{\frac{1}{2}}}J_{b-1}(2k^{\frac{1}{2}}x^{\frac{1}{2}}) - J_b(2k^{\frac{1}{2}}x^{\frac{1}{2}}\right]$$

By comparing the expansion of J_{b-1} and J_b we are able to prove the lemma. □

D.4 PROOF OF LEMMA 3.5.4

Proof. From Lemma (3.5.2), for (b^0, x^0), $\forall n$, $\exists\ p(n)$ such that $F(a_{p(n)}, b^0, x^0) = 0$, by the continuity of a_p with respect to (b, x), we know that there exists a neighborhood of (b^0, x^0) such that for $p \geq p(n)$, $-a_p > n$ in the closure. Let n large enough, we can make $\forall\epsilon > 0$, $k = \frac{b}{2} - a_{p(n)}$, then $\exists N$ such that

$$N\pi + \frac{\pi}{2} - \epsilon \leq w = 2k^{\frac{1}{2}}x^{\frac{1}{2}} - \frac{1}{2}b\pi + \frac{1}{4}\pi \leq N\pi + \frac{\pi}{2} + \epsilon$$

$$\Downarrow$$

$$(\frac{N\pi + \frac{\pi}{4} + \frac{1}{2}b\pi - \epsilon}{2x^{\frac{1}{2}}})^2 - \frac{b}{2} \leq -a_{p(n)} \leq (\frac{N\pi + \frac{\pi}{4} + \frac{1}{2}b\pi + \epsilon}{2x^{\frac{1}{2}}})^2 - \frac{b}{2} \qquad \text{(D.4.1)}$$

Because $-a_{p(n)}$ is large, N cannot be small. On the other hand, for $p(n)+m$ with m being positive integers, we have

$$\left(\frac{(N+m)\pi + \frac{\pi}{4} + \frac{1}{2}b\pi - \epsilon}{2x^{\frac{1}{2}}}\right)^2 - \frac{b}{2} \leq -a_{p(n)+m} \leq \left(\frac{(N+m)\pi + \frac{\pi}{4} + \frac{1}{2}b\pi + \epsilon}{2x^{\frac{1}{2}}}\right)^2 - \frac{b}{2} \quad \text{(D.4.2)}$$

Comparing Equation (D.3.1) and Equation (D.3.2), we can find c_1, c_2, l_1 and l_2 that satisfies Equation (3.5.4). To deal with the a_0 part: when p_n is large enough, we can have $-a_0(b^0, x^0) < c_1(p\pi - l_1)^2$, by continuity and compactness, we can find a neighbourhood of (b^0, x^0) such that $-a_0 < c_1(p\pi - l_1)^2$. Note that a_p is continuous function of b and x for each p. □

D.5 PROOF OF LEMMA 3.5.5

Proof. From Lemma (3.5.2), we know that

$$A_p(\alpha) = e^{(\alpha_3-\alpha_4)/2}\left(\frac{\alpha_3}{\alpha_4}\right)^{\frac{1-\alpha_2}{2}} \frac{(\pi\alpha_3^{\frac{1}{2}}k^{\frac{1}{2}})^{-\frac{1}{2}}\cos w_3 + R^2(s_p,\alpha_2,\alpha_3)}{\alpha_4^{\frac{1}{2}}k^{-\frac{1}{2}}(\pi\alpha_4^{\frac{1}{2}}k^{\frac{1}{2}})^{-\frac{1}{2}}\sin w_4 + R^3(s_p,\alpha_2,\alpha_4)} \tag{D.5.1}$$

where $k = \frac{\alpha_2}{2} - s_p$, $w_3 = 2k^{\frac{1}{2}}\alpha_3^{\frac{1}{2}} - \frac{1}{2}\alpha_2\pi + \frac{1}{4}\pi$, $w_4 = 2k^{\frac{1}{2}}\alpha_4^{\frac{1}{2}} - \frac{1}{2}\alpha_2\pi + \frac{1}{4}\pi$. On the other hand,

$$(\pi\alpha_4^{\frac{1}{2}}k^{\frac{1}{2}})^{-\frac{1}{2}}\cos w_4 + R^2(s_p,\alpha_2,\alpha_4) = 0$$

With the property of R^2 in Lemma (3.5.2) and Lemma (3.5.4), $\exists$ a neighborhood of (α_2^0,α_4^0), p_0, r_9 and r_{10} such that, when $p \geq p_0$, we have that in the neighborhood:

$$|\alpha_4^{\frac{1}{2}}k^{-\frac{1}{2}}(\pi\alpha_4^{\frac{1}{2}}k^{\frac{1}{2}})^{-\frac{1}{2}}\sin w_4 + R^3(s_p,\alpha_2,\alpha_4)| \geq r_9\alpha_4^{\frac{1}{4}}k^{-\frac{3}{4}}\pi^{-\frac{1}{2}},\ \text{and}$$
$$|(\pi\alpha_3^{\frac{1}{2}}k^{\frac{1}{2}})^{-\frac{1}{2}}\cos w_3 + R^2(s_p,\alpha_2,\alpha_3)| \leq r_{10}\alpha_3^{-\frac{1}{4}}k^{-\frac{1}{4}}\pi^{-\frac{1}{2}}.$$

So when $p \geq p_0$ large, in the closure, $\exists\, r_{11} > 0$ such that:

$$|A_p(\alpha)| \leq r_{11}k^{\frac{1}{2}} = r_{11}\sqrt{\frac{\alpha_2}{2} - s_p} \tag{D.5.2}$$

while $g(t|\alpha) = \sum_{p=0}^{\infty} A_p(\alpha)\exp(s_pt)$, consider $\frac{\sum_{p=1}^{p_0-1} A_p(\alpha)\exp(s_pt)}{A_0(\alpha)\exp(s_0t)}$ and $\frac{\sum_{p=p_0}^{\infty} A_p(\alpha)\exp(s_pt)}{A_0(\alpha)\exp(s_0t)}$ separately. By continuity and compactness, it is not hard to see that we can find $r_9^{(1)} < \frac{1}{2}$ and $t_1^{(1)}$ such that when $t \geq t_1^{(1)}$:

$$\frac{\sum_{p=1}^{p_0-1} |A_p(\alpha)|\exp(s_pt)}{A_0(\alpha)\exp(s_0t)} \leq r_9^{(1)} \tag{D.5.3}$$

By Equation (D.5.2) and Lemma (3.5.4), we have that:

$$\begin{aligned}\frac{\sum_{p=p_0}^{\infty} |A_p(\alpha)|\exp(s_pt)}{A_0(\alpha)\exp(s_0t)} &\leq \frac{\sum_{p=p_0}^{\infty} r_1\sqrt{\frac{\sup(\alpha_2)}{2} + c_2(p\pi - l_2)^2}\exp(-c_1(p\pi-l_2)^2t)}{A_0(\alpha)\exp(s_0t)} \\ &\leq \sum_{p=p_0}^{\infty} \frac{r_1\sqrt{\frac{\sup(\alpha_2)}{2} + c_2(p\pi - l_2)^2}}{\inf(A_0(\alpha))}\exp(-c_1(p\pi-l_2)^2t + \sup(-s_0)t)\end{aligned} \tag{D.5.4}$$

So $\exists\, t_1^{(2)}$ such that and a neighbourhood such that when $t \geq t_1^{(2)}$:

$$\frac{\sum_{p=p_0}^{\infty} |A_p(\alpha)|\exp(s_pt)}{A_0(\alpha)\exp(s_0t)} \leq r_9^{(2)} < \frac{1}{2} \tag{D.5.5}$$

Let $t_1 = \max(t_1^{(1)}, t_1^{(2)})$, then:

$$(1 - r_9^{(1)} - r_9^{(2)})A_0\exp(s_0t) \leq A_0(\alpha)\exp(s_0t)g(t|\alpha) \leq (1 + r_9^{(1)} + r_9^{(2)})A_0(\alpha)\exp(s_0t)$$

$\square$

D.6 PROOF OF LEMMA 3.5.6

Proof. From Equation (3.5.11) and the proof of Lemma (3.5.3), we can see that: With Equation (3.5.13), we know $\sum_{n=1}^{\infty} \frac{\partial^2 U_n}{\partial a^2}$ is uniform convergent and sum can be bounded by some term of order $\frac{1}{k}$. While $U_0 = J_{b-1}(2k^{\frac{1}{2}}x^{\frac{1}{2}}) + \frac{1}{2}b(\frac{x}{4k})J_{b+1}(2k^{\frac{1}{2}}x^{\frac{1}{2}})$, we can write Equation (3.5.14) as

$$\begin{aligned}\frac{\partial^2 F}{\partial a^2} &= \Gamma(b)e^{x/2}\left(\frac{1-b}{2}\right)\left(\frac{1-b}{2}-1\right)(kx)^{\frac{1-b}{2}-2}(x^2)\Big((\pi x^{\frac{1}{2}}k^{\frac{1}{2}})^{-\frac{1}{2}}\cos(w) + R^2(a,b,x)\Big) \\ &\quad + 2\Gamma(b)e^{x/2}\left(\frac{1-b}{2}\right)(kx)^{\frac{1-b}{2}-1}(-x)\Big(x^{\frac{1}{2}}k^{-\frac{1}{2}}(\pi x^{\frac{1}{2}}k^{\frac{1}{2}})^{-\frac{1}{2}}\sin(w) + R^3(a,b,x)\Big) \\ &\quad + \Gamma(b)e^{x/2}(kx)^{\frac{1}{2}-\frac{1}{2}b}\left(\frac{\partial^2 J_{b-1}(2k^{\frac{1}{2}}x^{\frac{1}{2}})}{\partial a^2} + R^5(a,b,x)\right)\end{aligned} \tag{D.6.1}$$

where when k large, $|R^5(a,b,x)| \leq k_5/k$. On the other hand,

$$\frac{\partial J_{b-1}(2k^{\frac{1}{2}}x^{\frac{1}{2}})}{\partial a} = J_b(2k^{\frac{1}{2}}x^{\frac{1}{2}})(k^{-\frac{1}{2}}x^{\frac{1}{2}}) - \frac{b-1}{2k}J_{b-1}(k^{\frac{1}{2}}x^{\frac{1}{2}})$$

and

$$\begin{aligned}\frac{\partial^2 J_{b-1}(2k^{\frac{1}{2}}x^{\frac{1}{2}})}{\partial a^2} =& \frac{1}{2}k^{-\frac{3}{2}}x^{\frac{1}{2}}J_b(2k^{\frac{1}{2}}x^{\frac{1}{2}}) \\ &+ (k^{-\frac{1}{2}}x^{\frac{1}{2}})\Big(-\frac{b}{2k^{\frac{1}{2}}x^{\frac{1}{2}}}J_b(2k^{\frac{1}{2}}x^{\frac{1}{2}}) + J_{b+1}(2k^{\frac{1}{2}}x^{\frac{1}{2}})\Big) \\ &+ \frac{b-1}{2k^2}J_{b-1}(k^{-\frac{1}{2}}x^{\frac{1}{2}}) \\ &- \frac{b-1}{2k}\Big(J_b(2k^{\frac{1}{2}}x^{\frac{1}{2}})(k^{\frac{1}{2}}x^{\frac{1}{2}}) - \frac{b-1}{2k}J_{b-1}(k^{\frac{1}{2}}x^{\frac{1}{2}})\Big)\end{aligned} \tag{D.6.2}$$

From the expansion of J_{b-1}, J_b and J_{b+1}, we know the leading term is $(k^{-\frac{1}{2}}x^{\frac{1}{2}})J_{b+1}(2k^{\frac{1}{2}}x^{\frac{1}{2}}) = -\sqrt{\frac{x^{\frac{1}{2}}}{\pi k^{\frac{3}{2}}}}\cos(w)(1 + \mathcal{O}(|k^{-\frac{1}{2}}|)))$. Combine all the observations above, we know Lemma (3.5.6) holds. □

D.7 PROOF OF LEMMA 3.5.7

Proof. From Equation (3.5.1), we know that we need to consider the behavior of $J_{b-1}(2x^{\frac{1}{2}}k^{\frac{1}{2}})$. First,

$$\begin{aligned} J_{b-1}(2x^{\frac{1}{2}}k^{\frac{1}{2}}) =&(\frac{1}{\pi x^{\frac{1}{2}}k^{\frac{1}{2}}})^{\frac{1}{2}}\Big(\big(\sum_{n=0}^{N-1}(-1)^n\frac{a_{2n}(b-1)}{(2x^{\frac{1}{2}}k^{\frac{1}{2}})^{2n}}+R_{2N}^{(J)}(2x^{\frac{1}{2}}k^{\frac{1}{2}},b-1)\big)\cos w \\ &\big(\sum_{m=0}^{M-1}(-1)^m\frac{a_{2m+1}(b-1)}{(2x^{\frac{1}{2}}k^{\frac{1}{2}})^{2m+1}}+R_{2M+1}^{(J)}(2x^{\frac{1}{2}}k^{\frac{1}{2}},b-1)\big)\sin w\Big) \end{aligned} \tag{D.7.1}$$

when $|b-1|<N+\frac{1}{2}$, we have $|R_N^{(J)}(2x^{\frac{1}{2}}k^{\frac{1}{2}},b-1)|<\frac{k_9}{(2x^{\frac{1}{2}}k^{\frac{1}{2}})^N}$.
Take N and M to be the smallest integer satisfies $2N>|b-1|-\frac{1}{2}$, $2M+1>|b-1|-\frac{1}{2}$, then we have $2N<|b-1|+\frac{3}{2}$, $2M+1<|b-1|+\frac{3}{2}$. Simply let

$$\begin{aligned} R_5 =&\big(\sum_{n=1}^{N-1}(-1)^n\frac{a_{2n}(b-1)}{(2x^{\frac{1}{2}}k^{\frac{1}{2}})^{2n}}+R_{2N}^{(J)}(2x^{\frac{1}{2}}k^{\frac{1}{2}},b-1)\big)\cos w \\ &\big(\sum_{m=0}^{M-1}(-1)^m\frac{a_{2m+1}(b-1)}{(2x^{\frac{1}{2}}k^{\frac{1}{2}})^{2m+1}}+R_{2M+1}^{(J)}(2x^{\frac{1}{2}}k^{\frac{1}{2}},b-1)\big)\sin w \end{aligned} \tag{D.7.2}$$

□

D.8 PROOF OF LEMMA 3.5.8

Proof. Lemma (3.5.3) and Lemma (3.5.7) naturally give the form and bound for R_1, R_5 and R_7, we only need to consider the expansion of $\frac{\partial J_{b-1}(2x^{\frac{1}{2}}k^{\frac{1}{2}})}{\partial(2x^{\frac{1}{2}}k^{\frac{1}{2}})}$:

$$\begin{aligned} \frac{\partial J_{b-1}(2x^{\frac{1}{2}}k^{\frac{1}{2}})}{\partial(2x^{\frac{1}{2}}k^{\frac{1}{2}})} =&-(\frac{1}{\pi x^{\frac{1}{2}}k^{\frac{1}{2}}})^{\frac{1}{2}}\Big(\big(\sum_{n=0}^{N-1}(-1)^n\frac{b_{2n}(b-1)}{(2x^{\frac{1}{2}}k^{\frac{1}{2}})^{2n}}+R_{2N}^{(J')}(2x^{\frac{1}{2}}k^{\frac{1}{2}},b-1)\big)\sin w \\ &\big(\sum_{m=0}^{M-1}(-1)^m\frac{b_{2m+1}(b-1)}{(2x^{\frac{1}{2}}k^{\frac{1}{2}})^{2m+1}}-R_{2M+1}^{(J')}(2x^{\frac{1}{2}}k^{\frac{1}{2}},b-1)\big)\cos w\Big) \end{aligned} \tag{D.8.1}$$

when $|b-1| < N - \frac{1}{2}$, we have $|R_N^{(J')}(2x^{\frac{1}{2}}k^{\frac{1}{2}}, b-1)| < \frac{k_{12}}{(2x^{\frac{1}{2}}k^{\frac{1}{2}})^N}$.

Similar with Lemma (3.5.7), take N and M to be the smallest integers satisfying $2N - \frac{1}{2} > |b-1|$, $2M + 1 - \frac{1}{2} > |b-1|$. Then $2N < |b-1| + \frac{5}{2}$, $2M + 1 < |b-1| + \frac{5}{2}$. Let

$$\begin{aligned} R^6(a,b,x) =& \Big(\sum_{n=1}^{N-1}(-1)^n \frac{b_{2n}(b-1)}{(2x^{\frac{1}{2}}k^{\frac{1}{2}})^{2n}} + R_{2N}^{(J')}(2x^{\frac{1}{2}}k^{\frac{1}{2}}, b-1)\Big)\sin w \\ & \Big(\sum_{m=0}^{M-1}(-1)^m \frac{b_{2m+1}(b-1)}{(2x^{\frac{1}{2}}k^{\frac{1}{2}})^{2m+1}} - R_{2M+1}^{(J')}(2x^{\frac{1}{2}}k^{\frac{1}{2}}, b-1)\Big)\cos w \end{aligned} \tag{D.8.2}$$

we have proved the lemma. □

D.9 PROOF OF LEMMA 3.5.9

Proof. Since α_1 is a time scaling parameter, it is not hard to see that $g(t|\alpha) = \sum_{p=0}^{+\infty} \frac{A_p(\alpha)}{\alpha_1} \exp(\frac{s_p}{\alpha_1}t)$. Then the genral case can be extended easily from the case $\alpha_1 = 1$. □

APPENDIX E

PROOF IN SECTION 3.6

E.1 PROOF OF THEOREM 3.6.1

Proof. If $\forall \rho > 0, \lim_n \rho x_n \log \mu_n(0, \frac{1}{\phi(\rho x_n)}) = -B$, by Lemma (3.6.3), we know that

$$\limsup_n \rho x_n \log M_n(\psi(\rho x_n)) \leq h(B).$$

However, $\forall \xi > 0, \liminf_n \rho x_n \log M_n(\psi(\rho x_n)) \geq -\xi^\alpha + \xi \liminf_n (\frac{\rho x_n}{\xi}) \log \mu_n(0, \frac{1}{\phi(\frac{\rho x_n}{\xi})})$, so we must have:

$$\liminf_n \rho x_n \log M_n(\psi(\rho x_n)) \geq h(B).$$

Then $\liminf_n \rho x_n \log M_n(\psi(\rho x_n)) = h(B)$.

If $\forall \rho > 0, \liminf_n \rho x_n \log M_n(\psi(\rho x_n)) = h(B)$. First, $\forall \xi > 0, \limsup_n \rho x_n \log M_n(\psi(\rho x_n)) \geq -\xi^\alpha + \xi \limsup_n (\frac{\rho x_n}{\xi}) \log \mu_n(0, \frac{1}{\phi(\frac{\rho x_n}{\xi})})$. Because of the arbitrary choice of ρ and ξ,

$$h(B) \geq -\xi^\alpha + \xi \limsup_n \rho x_n \log \mu_n(0, \frac{1}{\phi(\rho x_n)})$$

Then $\limsup_n \rho x_n \log \mu_n(0, \frac{1}{\phi(\rho x_n)}) \leq -B$. By Lemma (3.6.3) and (3.6.4), we know

$$\liminf_n \rho x_n \log \mu_n(0, \frac{1}{\phi(\rho x_n)}) \geq -B.$$

So we must have:

$$\lim_n \rho x_n \log \mu_n(0, \frac{1}{\phi(\rho x_n)}) = -B.$$

$\square$

E.2 PROOF OF LEMMA 3.6.2

Proof. We prove Inequality (3.6.2) for the $\xi > \lambda_0$ case. Let $\xi_k = \xi + k\epsilon$, $k = 0, 1, \cdots, K$. Assume $\frac{1}{\phi(\frac{\rho x_n}{\xi_k})} \geq \frac{1}{\phi(\frac{\rho x_n}{\xi_{k+1}})}$ then

$$\int_{\frac{1}{\phi(\frac{\rho x_n}{\xi_{k+1}})}}^{\frac{1}{\phi(\frac{\rho x_n}{\xi_k})}} \exp(-x\psi(\rho x_n))d\mu_n(x) \leq \mu_n(0, \frac{1}{\phi(\frac{\rho x_n}{\xi_k})}) \exp(-\frac{1}{\rho x_n} \times \frac{\phi(\rho x_n)}{\phi(\frac{\rho x_n}{\xi_{k+1}})}) \tag{E.2.1}$$

By Potter's theorem, when n large, $\forall k \in (-1, \cdots, K-1)$, $\exists \delta \in (0, 1)$ s.t.

$$\frac{\phi(\rho x_n)}{\phi(\frac{\rho x_n}{\xi_{k+1}})} \geq \delta \min(\xi_{k+1}^{\alpha'}, \xi_{k+1}^{\alpha''})$$

where $\alpha'' - \alpha = \alpha - \alpha' > 0$. In addition, by assumption, we can choose $-B' > -B$ s.t.

$$\mu_n(0, \frac{1}{\phi(\frac{\rho x_n}{\xi_k})}) \leq \exp(-\frac{1}{\rho x_n} B'\xi_k)$$

Inequality (E.2.1) becomes

$$\int_{\frac{1}{\phi(\frac{\rho x_n}{\xi_{k+1}})}}^{\frac{1}{\phi(\frac{\rho x_n}{\xi_k})}} \exp(-x\psi(\rho x_n))d\mu_n(x) \leq \exp(-\frac{1}{\rho x_n}(B'\xi_k + \delta \min(\xi_{k+1}^{\alpha'}, \xi_{k+1}^{\alpha''}))).$$

Let $p(x) = -B'x - \delta \min(x^{\alpha'}, x^{\alpha''})$, then

$$\int_{\frac{1}{\phi(\frac{\rho x_n}{\xi_{k+1}})}}^{\frac{1}{\phi(\frac{\rho x_n}{\xi_k})}} \exp(-x\psi(\rho x_n))d\mu_n(x) \leq \exp(\frac{1}{\rho x_n}(p(\xi_k) + B'\epsilon)).$$

When $x > \xi$,

$$\begin{aligned}\frac{p(x) - p(\xi)}{x - \xi} &= - B' - (\frac{\delta \min(x^{\alpha'}, x^{\alpha''}) - \delta \min(\xi^{\alpha'}, \xi^{\alpha''})}{x - \xi}) \\ &\leq - B' - \min(\alpha''\xi^{\alpha''-1}, \alpha'\xi^{\alpha'-1})\end{aligned}$$

Because $(-\alpha\lambda_0)^{\alpha-1} = B$, we know $(-\alpha\xi)^{\alpha-1} < B$. Then B', α' and α'' can be chosen s.t.

$$\max(-\alpha''\xi^{\alpha''-1}, -\alpha'\xi^{\alpha'-1}) < -B'.$$

Thus

$$\frac{p(x) - p(\xi)}{x - \xi} \leq -B' + \max(-\alpha''\xi^{\alpha''-1}, -\alpha'\xi^{\alpha'-1}) \leq C < 0.$$

So we can have:

$$\int_{\frac{1}{\phi(\frac{\rho x_n}{\xi_{k+1}})}}^{\frac{1}{\phi(\frac{\rho x_n}{\xi_k})}} \exp(-x\psi(\rho x_n))d\mu_n(x) \le \exp(\frac{1}{\rho x_n}(p(\xi)+C(k+1)\epsilon + B'\epsilon)).$$

Sum up the K terms:

$$\begin{aligned}\int_{\frac{1}{\phi(\frac{\rho x_n}{\xi_K})}}^{\frac{1}{\phi(\frac{\rho x_n}{\xi})}} \exp(-x\psi(\rho x_n))d\mu_n(x) &\le \frac{\exp(\frac{1}{\rho x_n}(p(\xi)+(B'+C)\epsilon))}{1-\exp(\frac{C\epsilon}{\rho x_n})} \\ &\le (1+o(1))\exp(\frac{1}{\rho x_n}(p(\xi)+B'\epsilon))\end{aligned}$$

On the other hand,

$$\int_{-\infty}^{\frac{1}{\phi(\frac{\rho x_n}{\xi_K})}} \exp(-x\psi(\rho x_n))d\mu_n(x) \le \mu_n(0, \frac{1}{\phi(\frac{\rho x_n}{\xi_K})}) \le \exp\left(\frac{1}{\rho x_n}(-B')\xi_K\right).$$

By making K large, we have:

$$\int_{-\infty}^{\frac{1}{\phi(\frac{\rho x_n}{\xi})}} \exp(-x\psi(\rho x_n))d\mu_n(x) \le (1+o(1))\exp(\frac{1}{\rho x_n}(p(\xi)+B'\epsilon)).$$

So

$$\limsup_n \rho x_n \log \int_{-\infty}^{\frac{1}{\phi(\frac{\rho x_n}{\xi})}} \exp(-x\psi(\rho x_n))d\mu_n(x) \le p(\xi)+B'\epsilon = -B'\xi - \delta\min(\xi^{\alpha'}, \xi^{\alpha''}).$$

Let $B' \uparrow B$, $\delta \uparrow 1$, $\alpha' \uparrow \alpha$, $\alpha'' \downarrow \alpha$ and $\epsilon \downarrow 0$, we know:

$$\limsup_n \rho x_n \log \int_{-\infty}^{\frac{1}{\phi(\frac{\rho x_n}{\xi})}} \exp(-x\psi(\rho x_n))d\mu_n(x) \le -B\xi - \xi^{\alpha}.$$

As to Inequality (3.6.1), use an analogues proof with $\xi_k = \xi - k\epsilon$. $\square$

E.3 PROOF OF LEMMA 3.6.3

Proof. With the same λ_0, choose $0 < \xi_1 < \lambda_0 < \xi_2 < +\infty$. Then

$$\begin{aligned}
&\limsup_n \rho x_n \log \int_{\frac{1}{\phi(\frac{\rho x_n}{\xi_2})}}^{\frac{1}{\phi(\frac{\rho x_n}{\xi_1})}} \exp(-x\psi(\rho x_n))d\mu_n(x) \\
\leq &\limsup_n \rho x_n \log(\exp(-\frac{\phi(\rho x_n)}{\rho x_n \phi(\frac{\rho x_n}{\xi_2})}) \times \mu_n(0, \frac{1}{\phi(\frac{\rho x_n}{\xi_1})})) \\
\leq &- B\xi_1 - \xi_2^\alpha
\end{aligned}$$

From Lemma (3.6.2),

$$\limsup_n \rho x_n \log \int_{\frac{1}{\phi(\frac{\rho x_n}{\xi_1})}}^{+\infty} e^{-x\psi(\rho x_n)} d\mu_n(x) \leq -B\xi_1 - \xi_1^\alpha < -B\xi_1 - \xi_2^\alpha$$

$$\limsup_n \rho x_n \log \int_{-\infty}^{\frac{1}{\phi(\frac{\rho x_n}{\xi_2})}} e^{-x\psi(\rho x_n)} d\mu_n(x) \leq -B\xi_2 - \xi_2^\alpha < -B\xi_1 - \xi_2^\alpha$$

Thus for large n

$$\int_{-\infty}^{+\infty} \exp(-x\psi(\rho x_n))d\mu_n(x) \leq 3\exp(-\frac{-B\xi_1 - \xi_2^\alpha}{\rho x_n})$$

Let $\xi_1 \uparrow \lambda_0$ and $\xi_2 \downarrow \lambda_0$, it is not hard to see:

$$\limsup_n \rho x_n \log(M_n(\psi(\rho x_n))) \leq h(B).$$

$\square$

E.4 PROOF OF LEMMA 3.6.4

Proof. From Lemma (3.6.3), $C \leq h(B) = \sup(-B\lambda - \lambda^{\alpha})$, so there are two roots $\lambda_1 \leq \lambda_2$ in $(0, +\infty)$, which coincides if and only if $h(B) = C$.

Choose $0 < \eta_1 < \lambda_1 \leq \lambda_2 < \eta_2 < \infty$, by Lemma (3.6.2):

$$\limsup_n \rho x_n \log \int_{\frac{1}{\phi(\frac{\rho x_n}{\eta_1})}}^{+\infty} e^{-x\psi(\rho x_n)} d\mu_n(x)$$

$$\leq - B\eta_1 - \eta_1^{\alpha} < -B\lambda_1 - \lambda_1^{\alpha} = C$$

$$\limsup_n \rho x_n \log \int_{-\infty}^{\frac{1}{\phi(\frac{\rho x_n}{\eta_2})}} e^{-x\psi(\rho x_n)} d\mu_n(x)$$

$$\leq - B\eta_2 - \eta_2^{\alpha} < -B\lambda_2 - \lambda_2^{\alpha} = C$$

So for ϵ small and n large, we know:

$$\int_{\frac{1}{\phi(\frac{\rho x_n}{\eta_1})}}^{+\infty} e^{-x\psi(\rho x_n)} d\mu_n(x) \leq \exp(\frac{1}{\rho x_n}(C - 2\epsilon))$$

$$\int_{-\infty}^{\frac{1}{\phi(\frac{\rho x_n}{\eta_2})}} e^{-x\psi(\rho x_n)} d\mu_n(x) \leq \exp(\frac{1}{\rho x_n}(C - 2\epsilon))$$

By hypothesis, we have: when n is large

$$\int_{-\infty}^{+\infty} e^{-\psi(\rho x_n)x} d\mu_n(x) \geq \exp(\rho x_n(C - \epsilon))$$

Then $\int_{\frac{1}{\phi(\frac{\rho x_n}{\eta_2})}}^{\frac{1}{\phi(\frac{\rho x_n}{\eta_1})}} e^{-\psi(\rho x_n)x} d\mu_n(x) \geq (1 + o(1)) \exp(\rho x_n(C - \epsilon))$. By which we have:

$$\liminf_n \rho x_n \log \int_{\frac{1}{\phi(\frac{\rho x_n}{\eta_2})}}^{\frac{1}{\phi(\frac{\rho x_n}{\eta_1})}} e^{-\psi(\rho x_n)x} d\mu_n(x) \geq C$$

On the other hand,

$$\liminf_n \rho x_n \log \int_{\frac{1}{\phi(\frac{\rho x_n}{\eta_2})}}^{\frac{1}{\phi(\frac{\rho x_n}{\eta_1})}} e^{-\psi(\rho x_n)x} d\mu_n(x)$$

$$\leq \liminf_n \rho x_n \log(\exp(-\frac{\phi(\rho x_n)}{\phi(\frac{\rho x_n}{\eta_2})} \frac{1}{\rho x_n}) \times \mu_n(0, \frac{1}{\phi(\frac{\rho x_n}{\eta_1})}))$$

$$\leq - \eta_2^{\alpha} + \eta_1 \liminf_n \frac{\rho x_n}{\eta_1} l \log(\mu_n(0, \frac{1}{\phi(\frac{\rho x_n}{\eta_1})}))$$

So $\liminf_n \frac{\rho x_n}{\eta_1} l \log(\mu_n(0, \frac{1}{\phi(\frac{\rho x_n}{\eta_1})})) \geq \frac{C + \eta_2^{\alpha}}{\eta_1}$. Because of the arbitrary choice of ρ:

$$\liminf_n \rho x_n \log(\mu_n(0, \frac{1}{\rho x_n})) \geq \frac{C + \eta_2^{\alpha}}{\eta_1}$$

Let $\eta_2 \downarrow \lambda_2$, $\eta_1 \uparrow \lambda_1$, then we finish the proof. $\square$

E.5 PROOF OF LEMMA 3.6.5

Proof. We prove the lemma by contradiction. In a small closed neighbourhood of α^0, if $l(t|\alpha) \not\to 0$ uniformly. Then $\exists\ \epsilon > 0$, $\alpha^n \to \tilde{\alpha}$ are in the neighbourhood, $t_n \to 0$ s.t. $|l(\alpha^n, t_n|)| > \epsilon$.

First from the Laplace transform of $g(t|\alpha)$, we know that $g(t|\alpha)$ is differentialbe with respect to t, and we have:

$$\int_0^{+\infty} \exp(-st)\frac{\partial g(t|\alpha)}{\partial t}dt = \frac{sF(s\alpha_1, \alpha_2, \alpha_3)}{F(s\alpha_1, \alpha_2, \alpha_4)}.$$

By Theorem 1.2 in (Rosler, 1980), the density function of first passage time of diffusion is unimodal (actually we can prove this argument by constructing a sequence of birth-and-death processes that converges to the Cox-Ingersoll-Ross model weakly). Then $\exists\ t_0$ such that, when $0 < t \leq t_0$, $\frac{\partial g(t|\tilde{\alpha})}{\partial t} > 0$. By the continuity of $\frac{\partial g(t|\alpha)}{\partial t}|_{t=t_0}$ in α, we know that $\exists\ N$ such that, when $n > N$, $\frac{\partial g(t|\alpha^n)}{\partial t}|_{t=t_0} > 0$. By unimodality again, we know that if $n > N$, $t \leq t_0$, $\frac{\partial g(t|\alpha^n)}{\partial t} > 0$. So we can treat $g(t|\alpha^n)$ as a sequence of measures with radon nikodym derivative $\frac{\partial g(t|\alpha^n)}{\partial t}$ on $(0, t_0)$.

On the other hand, from Equation (3.5.5), Lemma (3.5.4) and Inequality (D.5.2), we know that $\exists\ k$ and N_1 such that when $n > N_1$, $t > t_0$

$$|\frac{\partial g(t|\alpha^n)}{\partial t}| \leq k.$$

By which we know:

$$\frac{sF(s\alpha_1, \alpha_2, \alpha_3)}{F(s\alpha_1, \alpha_2, \alpha_4)} - k\frac{\exp(-st_0)}{s} \leq \int_0^{t_0} \exp(-st)\frac{\partial g(t|\alpha^n)}{\partial t}dt \leq \frac{sF(s\alpha_1, \alpha_2, \alpha_3)}{F(s\alpha_1, \alpha_2, \alpha_4)} + k\frac{\exp(-st_0)}{s}. \tag{E.5.1}$$

When $s \to \infty$, $\frac{sF(s\alpha_1,\alpha_2,\alpha_3)}{F(s\alpha_1,\alpha_2,\alpha_4)}$ dominates $k\frac{\exp(-st_0)}{s}$, then by Lemma (3.4.1), $\forall\ \rho > 0$, take $s = \frac{1}{t_n}$:

$$\lim_n \rho t_n \log \int_0^{t_0} \exp(-\frac{t}{\rho^2 t_n^2})\frac{\partial g(t|\alpha^n)}{\partial t}dt = -2\sqrt{\tilde{\alpha}_1}(\sqrt{\tilde{\alpha}_4} - \sqrt{\tilde{\alpha}_3}) \tag{E.5.2}$$

With Theorem (3.6.1), take $\phi(x) = \frac{1}{x}$, $\psi(x) = \frac{1}{x^2}$, we know that:

$$\lim_n t_n \log g(t_n|\alpha_n) = -\tilde{\alpha}_1(\sqrt{\tilde{\alpha}_4} - \sqrt{\tilde{\alpha}_3})^2 \tag{E.5.3}$$

which contradicts with our assumption. $\square$

APPENDIX F

PROOF IN SECTION 3.7

F.1 PROOF OF LEMMA 3.7.1

Proof. Define $m_\alpha(t) = \log g(t|\alpha)$, we first show that $\mathcal{M} = \{m_\alpha : ||\alpha - \alpha^0|| \le \delta, \alpha \in \mathcal{A}\}$ is a both a Glivenko-Cantelli class and a Donsker class under probability measure P_{α^0} (the measure with density function $g(t|\alpha)$). By Lemma (3.6.5), $\exists\ A_1 > 0$, $A_2 > 0$ such that $\forall \epsilon > 0$, $\forall K > 0$, $\forall m_\alpha \in \mathcal{M}$, $0 < t \le K\epsilon$:

$$m_\alpha^2(t) \le \frac{A_1^2}{t^2}, \quad g(t|\alpha^0) \le \exp(-\frac{A_2}{t})$$

by which we have $\forall m_\alpha, m_\beta \in \mathcal{M}$:

$$\begin{aligned}\int_0^{K\epsilon} (m_\alpha(t) - m_\beta(t))^2 g(t|\alpha^0)dt &\le \int_0^{K\epsilon} \frac{4A_1^2}{t^2} \exp(-\frac{A_2}{t})dt \\ &= \int_{\frac{1}{K\epsilon}}^{+\infty} 4A_1^2 \exp(-A_2 t)dt \\ &= 4\frac{A_1^2}{A_2} \exp(-\frac{A_2}{K\epsilon})\end{aligned}$$

So we can find a universal $K_1 > 0$, such that $\forall \epsilon > 0, \forall m_\alpha \in \mathcal{M}$,

$$\int_0^{K_1\epsilon} (m_\alpha(t) - m_\beta(t))^2 g(t|\alpha^0)dt \le \frac{\epsilon^2}{2} \tag{F.1.1}$$

So when $0 < t \le K_1\epsilon$, we can choose the only one bracket to be $[-\frac{A_1}{t}, \frac{A_1}{t}]$, which can cover $m_\alpha(t)$. When $K_1\epsilon < t < t_1$ where t_1 is the same one in Lemma (3.5.9), by the continuity of

$\frac{\partial \log g(t|\alpha)}{\partial \alpha} = \frac{\frac{\partial g(t|\alpha)}{\partial \alpha}}{g(t|\alpha)}$, $\exists A_3 > 0, A_4 > 0$:

$$\begin{aligned} |m_\alpha(t) - m_\beta(t)| &\leq \frac{A_3}{\exp(\frac{-A_4}{t})} ||\alpha - \beta|| \\ &\leq A_3 \exp(\frac{A_4}{K_1 \epsilon}) ||\alpha - \beta|| \end{aligned} \tag{F.1.2}$$

When $t \geq t_1$, $\exists A_5 > 0, A_6 > 0$ such that:

$$|m_\alpha(t) - m_\beta(t)| \leq (A_5 + A_6 t) ||\alpha - \beta|| \tag{F.1.3}$$

Define

$$m(t) = \begin{cases} A_3 \exp(\frac{A_4}{t}) & \text{if } K_1 \epsilon < t < t_1 \\ (A_5 + A_6 t) & \text{if } t \leq t_1 \end{cases} \tag{F.1.4}$$

Then we can cover the class of functions using a class of brackets $\{[m_\alpha - \frac{A_7 \epsilon m}{\sqrt{2}}, m_\alpha + \frac{A_7 \epsilon m}{\sqrt{2}}] : ||\alpha - \alpha^0|| \leq \delta\}$. From Equation (F.1.2) and (F.1.3), the number of brackets we need is at most $(\frac{\sqrt{2}\delta}{A_7 \epsilon})^4$. Now we give an value for A_7:

$$\begin{aligned} \int_{K_1\epsilon}^{+\infty} 2A_7^2 \epsilon^2 m^2(t) g(t|\alpha^0) dt &\leq \int_{K_1\epsilon}^{t_1} 2A_7^2 \epsilon^2 A_3^2 \exp(\frac{2A_4}{t}) g(t|\alpha^0) dt + \int_{t_1}^{\infty} 2A_7^2 \epsilon^2 (A_5 + A_6 t)^2 g(t|\alpha^0) dt \\ &\leq \int_{K_1\epsilon}^{t_1} 2A_7^2 \epsilon^2 A_3^2 \exp(\frac{2A_4}{K_1 \epsilon}) g(t|\alpha^0) dt + \int_{t_1}^{\infty} 2A_7^2 \epsilon^2 (A_5 + A_6 t)^2 g(t|\alpha^0) dt \\ &\leq A_7^2 \epsilon^2 \exp(\frac{2A_4}{K_1 \epsilon}) C_1 \end{aligned}$$

Since we confine ourselves with $\int_{K_1\epsilon}^{+\infty} 2A_7^2 \epsilon^2 m^2(t) g(t|\alpha^0) dt \leq \frac{\epsilon^2}{2}$, thus: $A_7 \leq C \exp(-\frac{A_4}{K_1 \epsilon})$. The number of brackets can be bounded by $(\frac{\sqrt{2}\delta}{C\epsilon} \exp(\frac{A_4}{K_1 \epsilon}))^4$. Combine the two parts, using the notation in (Van der Vaart, 2000):

$$N_{[\ ]}(\epsilon, \mathcal{M}, L^2(P_{\alpha^0})) \leq (\frac{\sqrt{2}\delta}{C\epsilon} \exp(\frac{A_4}{K_1 \epsilon}))^4 \tag{F.1.5}$$

While $J_{[\ ]}(\xi, \mathcal{M}, L^2(P_{\alpha^0})) = \int_0^\xi \sqrt{\log N_{[\ ]}(\xi, \mathcal{M}, L^2(P_{\alpha^0}))} d\xi$, we know $J_{[\ ]}(\xi, \mathcal{M}, L^2(P_{\alpha^0})) < +\infty$, by Theorem (19.5) in (Van der Vaart, 2000), $\mathcal{M}$ is Donsker. Actually, we only need that $\mathcal{M}$ is a Glivenko-Cantelli class, which is obviously true from the proof. Apply Theorem (5.7) in (Van der Vaart, 2000) to the log likelihood function, with compactness of $\mathcal{K}$, the MLE $\hat{\alpha}_n$ converges in probability to the true parameter α^0. $\square$

F.2 PROOF OF THEOREM 3.7.2

Proof. By a similar method used to prove Lemma (3.3.1), given only the observations, the identifiable parameters are α. With the definition of $\tilde{g}(t|\alpha,\Delta)$, we know that:

$$\frac{\partial \log \tilde{g}(t|\alpha,\Delta)}{\partial \alpha_i} = \frac{g_i(t|\alpha)}{g(t|\alpha)} + \frac{\int_0^\Delta g_i(s|\alpha)ds}{1-\int_0^\Delta g(s|\alpha)ds} \tag{F.2.1}$$

Because the support of t is bounded away from zero, by the continuity of $\frac{\partial \log \tilde{g}(t|\alpha,\Delta)}{\partial \alpha_i}$, Lemma (3.5.9) and the mean value theorem for the multivariate case, we know that $\forall \alpha$ and β in a neighbourhood of the true parameter α_0, $\exists\, q(t) = a + bt$ such that:

$$\left|\frac{\partial \log \tilde{g}(t|\alpha,\Delta)}{\partial \alpha_i}\right| \leq q(t) \quad \text{for} \quad t \geq \Delta \tag{F.2.2}$$

$$|\log \tilde{g}(t|\alpha,\Delta) - \log \tilde{g}(t|\beta,\Delta)| \leq q(t) \quad \text{for} \quad t \geq \Delta \tag{F.2.3}$$

By dominated convergence theorem, the information matrix $\tilde{I}(\alpha)$ is continuous at α^0. Using Theorem (7.6) in (Van der Vaart, 2000), $\alpha \to \sqrt{\tilde{g}(t|\alpha,\Delta)}$ is differentiable in quadratic mean. From Equation (F.2.3), the class of log of density functions also satisfies Lipschitz condition. With Example (19.7) in (Van der Vaart, 2000), it is a Glivenko-Cantelli class. Applying Theorem (5.7) in (Van der Vaart, 2000), we know that the consistency is valid. By Lipschitz condition along with differentiability in quadratic mean, we can apply Theorem (5.49) in (Van der Vaart, 2000) to have asymptotic normality. □

APPENDIX G

PROOF IN SECTION 5.1

G.1 PROOF OF LEMMA 5.1.1

Proof. First we consider $M_t = f(Y_t)e^{ct}$ and by Ito's lemma, we have that

$$dM_t = f(Y_t)ce^{ct}dt + e^{ct}\nabla U \cdot dY_t + e^{ct}\frac{1}{2}(dY_t)'HdY_t \tag{G.1.1}$$

where H is the Hessian matrix of U. Thus

$$dM_t = e^{ct}\left[\left(\frac{1}{2}\Delta f + cf - \nabla f \cdot Y\right)dt + \nabla f \cdot dB_t,\right] \tag{G.1.2}$$

so that $M_{t\wedge\tau} = f(Y_{t\wedge\tau})e^{c(t\wedge\tau)}$ is a local martingale. On the other hand, if we fix t, then $M_{s\wedge t\wedge\tau}$ is again a local martingale. Because f is bounded and e^{cs} is bounded on $s < t$, we know that $M_{s\wedge t\wedge\tau}$ is a martingale. Thus, we have that

$$E_x M_{s\wedge t\wedge\tau} = E_x M_0 = f(x) \tag{G.1.3}$$

Let $s \uparrow \tau$, by the dominated convergence theorem

$$E_x M_{\tau\wedge t} = f(x) \tag{G.1.4}$$

On the other hand,

$$E_x M_{\tau\wedge t} = E_x(f(Y_\tau)e^{c\tau}, \tau \le t) + E_x(f(Y_t)e^{ct}, \tau > t) \tag{G.1.5}$$

For the first part, $E_x e^{c\tau} < \infty$ provided the moment generating function exists for τ_1, τ_2, $P(\tau < \infty) = 1$, by the dominated convergence theorem,

$$E_x(f(Y_\tau)e^{c\tau}, \tau \leq t) \to E_x(f(Y_\tau)e^{c\tau}) = E_x(e^{c\tau}) \; as \; t \to \infty \tag{G.1.6}$$

For the second part, if $c \leq 0$, then $e^{ct} \leq 1$, so by dominated convergence theorem, $E_x(f(Y_t)e^{ct}, \tau > t) \to 0$ as $t \to \infty$. And if if $c > 0$, $f(Y_t)e^{ct} \leq f(Y_t)e^{\tau}$ on $\tau > t$, and by dominated convergence theorem again,

$$E_x(f(Y_t)e^{ct}, \tau > t) \to 0 \tag{G.1.7}$$

In conclusion, we have

$$f(x) = E_x(e^{c\tau}). \tag{G.1.8}$$

$\square$

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